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Susumu Ariki, Nicolas Jacon, Cédric Lecouvey

THE MODULAR BRANCHING RULE FOR AFFINE HECKE ALGEBRAS OF TYPE A

SUSUMU ARIKI, NICOLAS JACON AND CÉDRIC LECOUVEY

Abstract. For the affine Hecke algebra of type A at roots of unity, we make explicit the correspondence between geometrically constructed simple modules and combinatorially constructed simple modules and prove the modular branching rule. The latter generalizes work by Vazirani.

1. Introduction

In [6], Ginzburg explains his geometric construction of simple modules over (extended) affine Hecke algebras H_n defined over $\mathbb{C}$. In this paper, we consider the affine Hecke algebra of type A whose parameter is a root of unity. Then, the simple modules are labelled by aperiodic multisegments.

On the other hand, Dipper, James and Mathas' Specht module theory gives us a combinatorial construction of simple modules of cyclotomic Hecke algebras, and they exhaust all the simple modules of the affine Hecke algebra. The simple modules are labelled by Kleshchev multipartitions.

If one wants to compute something about simples, the combinatorially defined simple modules often have more advantage than the geometrically defined simple modules, and we may work over any algebraically closed field. On the other hand, the geometrically defined simple modules are very useful in several circumstances. Hence, explicit description of the module correspondence between the two constructions is desirable.

We provide this explicit description of the module correspondence in this article. Note that both the set of aperiodic multisegments and the set of Kleshchev multipartitions have structure of Kashiwara crystals. Then, we show that the crystal embedding gives the module correspondence. We also describe the crystal embedding explicitly.

Closely related to this result is the modular branching rule. One may prove the result on the module correspondence by using this, which is our first proof, or one may prove the modular branching rule by first establishing the result on the module correspondence, which is our second proof. Note that we mean here the modular branching rule in the original sense as we explain in the next paragraph. Some authors use the terminology in weaker sense, which does not imply the module correspondence.

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Let L_ψ be the simple module labelled by a multisegment ψ, whose precise meaning will be explained in section 4. The *modular branching rule* is a rule to describe $\mathrm{Soc}(i\text{-}\mathrm{Res}^{H_n}_{H_{n-1}}(L_\psi))$, or equivalently $\mathrm{Top}(i\text{-}\mathrm{Res}^{H_n}_{H_{n-1}}(L_\psi))$. We show that

$$\mathrm{Soc}(i\text{-}\mathrm{Res}^{H_n}_{H_{n-1}}(L_\psi)) = L_{\tilde{e}_i\psi},$$

where $\tilde{e}_i$ is the Kashiwara operator. We give a geometric proof of this rule in the framework of Lusztig and Ginzburg's theory. This gives the first proof. On the other hand, if one uses results in [2] and [3], both become easier, and this is the second proof.

Recall that the main result of [27] is the modular branching rule when the parameter of the affine Hecke algebra is not a root of unity. Hence our result generalizes [27, Theorem 3.1]. In [1] and [5] it was proved that affine sl_e controls the modular representation theory of cyclotomic Hecke algebras. Later[1], Grojnowski gave another proof [8, Theorem 14.2, 14.3] for several main results in [5]. We note here that he writes in [8, 14.1] that his IMRN paper proved the part concerning the canonical basis in [1], but his announcement in 1995 was that he had some computation of Kazhdan-Lusztig polynomials which he was able to deduce from the IMRN paper: the use of i-induction and i-restriction functors, integrable modules over $\mathfrak{g}(A^{(1)}_{e-1})$, and Lusztig's aperiodicity were absent in the assertion.

The idea of his proof in [8] came from Leclerc's observation that Kleshchev and Brundan's work on the modular branching rule of the symmetric group and the Hecke algebra of type A may be understood in crystal language. The proof is interesting, but the modular branching rule in our sense is not proved in [8] and it is natural to ask whether the crystal he used coincides with the one used in [1] and [5]. It was settled affirmatively in [3], but it still used several results from [8]. Here in this paper, the modular branching rule for the affine Hecke algebra, which is a stronger statement than the statement in [8] that the modular branching gives a crystal which is isomorphic to $B(\infty)$, is proved in a direct manner. It still uses the multiplicity one result from [9], but it replaces [8].

The paper is organized as follows. In section 2, we review basic facts on the crystal $B(\infty)$ of type $A^{(1)}_{e-1}$. In section 3, we prepare for a geometric proof of the modular branching rule of the affine Hecke algebra. In section 4, we give the geometric proof of the modular branching rule. In section 5, we introduce crystals of deformed Fock spaces and state results to compute crystal isomorphisms among them. In section 6, we prove a lemma on the module correspondence of simple modules in various labellings and give a combinatorial proof of the modular branching rule in the framework of Fock space theory for cyclotomic Hecke algebras.

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[1] Compare the submission date of [5] with [8].

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2. Preliminaries

Let $e \geq 2$ be a fixed integer, $\mathfrak{g}$ the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$. We denote by U_v^- the negative part of the quantum affine algebra $U_v(\mathfrak{g})$, which is generated by the Chevalley generators f_i, where $i \in \mathbb{Z}/e\mathbb{Z}$, subject to the quantum Serre relations. In this section, we review basic facts on U_v^- and its crystal. We denote the simple roots by α_i, and the simple coroots by $\alpha_i^\vee$, for $i \in \mathbb{Z}/e\mathbb{Z}$.

2.1. **The crystal $B(\infty)$.** Let us introduce the Kashiwara operator $\tilde{f}_i$, for $i \in \mathbb{Z}/e\mathbb{Z}$, on U_v^-. Let e_i, $i \in \mathbb{Z}/e\mathbb{Z}$, be Chevalley generators of the positive part of $U_v(\mathfrak{g})$ and $t_i = v^{\alpha_i^\vee}$. The following two lemmas are due to Kashiwara.

Lemma 2.1. *For each $u \in U_v^-$, there exist unique u' and u'' in U_v^- such that we have*

$$e_i u - u e_i = \frac{t_i u' - t_i^{-1} u''}{v - v^{-1}}.$$

We define an operator e_i' on U_v^- by $e_i' u = u''$, for $u \in U_v^-$. The algebra generated by $\{f_i\}_{i \in \mathbb{Z}/e\mathbb{Z}}$ and $\{e_i'\}_{i \in \mathbb{Z}/e\mathbb{Z}}$ is called the *Kashiwara algebra.* Let $f_i^{(n)}$ be the n^{th} divided power of f_i.

Lemma 2.2. *Let $P \in U_v^-$. For each $i \in \mathbb{Z}/e\mathbb{Z}$, there exists u_n in U_v^-, for $n \in \mathbb{Z}_{\geq 0}$, such that $e_i' u_n = 0$, for all n, and $P = \sum_{n \in \mathbb{Z}_{\geq 0}} f_i^{(n)} u_n$.*

We define $\tilde{e}_i P = \sum_{n \in \mathbb{Z}_{\geq 1}} f_i^{(n-1)} u_n$ and $\tilde{f}_i P = \sum_{n \in \mathbb{Z}_{\geq 0}} f_i^{(n+1)} u_n$. They are well-defined. Let R be the subring of $\mathbb{C}(v)$ consisting of elements which are regular at $v = 0$. Then, we define

$$L(\infty) = \sum_{N \in \mathbb{Z}_{\geq 0}} \sum_{(i_1, \ldots, i_N) \in (\mathbb{Z}/e\mathbb{Z})^N} R \tilde{f}_{i_1} \cdots \tilde{f}_{i_N} 1$$

and

$$B(\infty) = \Big(\cup_{N \in \mathbb{Z}_{\geq 0}} \cup_{(i_1, \ldots, i_N) \in (\mathbb{Z}/e\mathbb{Z})^N} \tilde{f}_{i_1} \cdots \tilde{f}_{i_N} 1 + vL(\infty) \Big) \setminus \{0\}.$$

$B(\infty)$ is a basis of the $\mathbb{C}$-vector space $L(\infty)/vL(\infty)$. U_v^- admits a root space decomposition $U_v^- = \oplus_{\alpha \in Q_+} (U_v^-)_{-\alpha}$, where $Q_+ = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{Z}_{\geq 0} \alpha_i$, and it follows that

$$B(\infty) = \bigsqcup_{\alpha \in Q_+} B(\infty)_{-\alpha}.$$

We define $\mathrm{wt}(b) = -\alpha$ if $b \in B(\infty)_{-\alpha}$. Then, by defining

$$\epsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\} \quad \text{and} \quad \varphi_i(b) = \epsilon_i(b) + \mathrm{wt}(b)(\alpha_i^\vee),$$

for $b \in B(\infty)$, $(B(\infty), \mathrm{wt}, \epsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is a $\mathfrak{g}$-crystal in the sense of Kashiwara [14, p.48].

We define the bar operation on U_v^- by $\bar{v} = v^{-1}$ and $\bar{f}_i = f_i$. Lusztig and Kashiwara independently constructed the canonical basis/the global basis

$$\{G_v(b) \mid b \in B(\infty)\}$$

of U_v^-, which is characterized by the property that

$$\overline{G_v(b)} = G_v(b), \quad G_v(b) + vL(\infty) = b.$$

Example 2.3. Let $e = 3$. Then, e_2 and f_1 commute so that $e_2' f_1 = 0$ and $\tilde{f}_2 f_1 = f_2 f_1$ follows. Similarly, $\tilde{f}_1 f_2 = f_1 f_2$. Thus, $\{f_1 f_2, f_2 f_1\}$ is the canonical basis of $(U_v^-)_{-\alpha_1-\alpha_2}$. For the null root $\delta = \alpha_0 + \alpha_1 + \alpha_2$, $\{f_0 f_1 f_2, f_2 f_1 f_0, f_0 f_2 f_1, f_1 f_0 f_2\}$ is the canonical basis of $(U_v^-)_{-\delta}$. Of course, more complex linear combination of monomials in f_i appear in the canonical basis of other $(U_v^-)_{-\alpha}$.

2.2. **Hall algebras.** The crystal $B(\infty)$ has a concrete description. Let Γ be the cyclic quiver of length e. This is an oriented graph with vertices $\mathbb{Z}/e\mathbb{Z}$ and edges $\{(i, i+1), i \in \mathbb{Z}/e\mathbb{Z}\}$. Let $V = \oplus_{i \in \mathbb{Z}/e\mathbb{Z}} V_i$ be a finite dimensional $\mathbb{Z}/e\mathbb{Z}$-graded vector space, and define

$$E_V = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(V_i, V_{i+1}) \subseteq \mathrm{End}_{\mathbb{C}}(V).$$

An element $X \in E_V$ is called a *representation* of Γ on V. The vector

$$\underline{\dim}\, V = (\dim V_i)_{i \in \mathbb{Z}/e\mathbb{Z}}$$

is called the *dimension vector* of the representation.

If V runs through all finite dimensional $\mathbb{Z}/e\mathbb{Z}$-graded vector spaces, we obtain the category of representations of Γ. It is the same as the category of finite dimensional $\mathbb{C}\Gamma$-modules, where $\mathbb{C}\Gamma$ is the path algebra of Γ. If X is nilpotent as an endomorphism of V, we say that the representation X, or the corresponding $\mathbb{C}\Gamma$-module, is *nilpotent*. We denote by $\mathcal{N}_V$ the subset of nilpotent representations in E_V. Let $G_V = \prod_{i \in \mathbb{Z}/e\mathbb{Z}} \mathrm{GL}(V_i)$. It acts on E_V and $\mathcal{N}_V$ by conjugation and two representations are equivalent if and only if they are in the same G_V-orbit.

For each $i \in \mathbb{Z}/e\mathbb{Z}$, let $V = V_i = \mathbb{C}$ and $X = 0$. Then it defines a simple $\mathbb{C}\Gamma$-module. We denote it by S_i. They are nilpotent representations.

Example 2.4. Let $G_n = \mathrm{GL}_n(\mathbb{C})$ and suppose that $s \in G_n$ has order e. Let ζ be a primitive e^{th} root of unity, $V = \mathbb{C}^n$, and let V_i be the eigenspace of s for the eigenvalue ζ^i. If $X \in \mathrm{End}_{\mathbb{C}}(V)$ is such that $sXs^{-1} = \zeta X$ then $XV_i \subseteq V_{i+1}$. Thus, X defines a representation of Γ on V. Note that G_V is the centralizer group $G_n(s)$ in this case.

By linear algebra, the isomorphism classes of nilpotent representations are labelled by ($\mathbb{Z}/e\mathbb{Z}$-valued) multisegments.

Definition 2.5. Let $l \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}/e\mathbb{Z}$. The *segment of length l and head i* is the sequence of consecutive residues $[i, i+1, ..., i+l-1]$. We denote it by $[i; l)$. Similarly, The *segment of length l and tail i* is the sequence of consecutive residues $[i-l+1, ..., i-1, i]$. We denote it by $(l; i]$. We say that $[i; l)$ has a *removable i-node* and $[i+1; l)$ has an *addable i-node.*

A collection of segments is called a *multisegment.* If the collection is the empty set, we call it the empty multisegment.

Each $[i; l)$ defines an indecomposable nilpotent $\mathbb{C}\Gamma$-module $\mathbb{C}[i; \ell)$, which is characterized by the property that

$$\mathbb{C}[i; l) \text{ is a uniserial module and } \mathrm{Top}(\mathbb{C}[i; l)) = S_i.$$

Hence, a complete set of isomorphism classes of nilpotent representations is given by the modules

$$M_\psi = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{Z}_{>0}} \mathbb{C}[i, l)^{\oplus m_{[i;l)}},$$

which is labelled by the multisegment

$$\psi = \{[i; l)^{\oplus m_{[i;l)}}\}_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{Z}_{>0}}.$$

We denote the corresponding G_V-orbit in $\mathcal{N}_V$ by $\mathcal{O}_\psi$.

Now, we introduce the Hall polynomials. Let $\mathbb{F}_q$ be a finite field, and consider $\mathbb{F}_q\Gamma$-modules. Then, they are classified by multisegments again. Let V, T and W be $\mathbb{Z}/e\mathbb{Z}$-graded vector spaces over $\mathbb{F}_q$ such that

$$\underline{\dim}\, V = \underline{\dim}\, T + \underline{\dim}\, W.$$

Let φ_1, φ_2 and ψ be multisegments such that $\mathcal{O}_{\varphi_1} \subseteq \mathcal{N}_T$, $\mathcal{O}_{\varphi_2} \subseteq \mathcal{N}_W$ and $\mathcal{O}_\psi \subseteq \mathcal{N}_V$. If the number of submodules U of M_ψ that satisfies $U \simeq M_{\varphi_2}$ and $M_\psi/U \simeq M_{\varphi_1}$ is polynomial in $q = \mathrm{card}(\mathbb{F}_q)$, then this polynomial is called the *Hall polynomial* and we denote it by $F^\psi_{\varphi_1,\varphi_2}(q)$. The existence of Hall polynomials in our case was proved by Jin Yun Guo [10, Theorem 2.7].

For a and b in $\mathbb{Z}^e$ we define a bilinear form m by

$$m(a, b) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (a_i b_{i+1} + a_i b_i).$$

We remark that this is not the Euler form used by Ringel to define his (twisted) Hall algebra, but the one used by Lusztig, which comes from the difference of dimensions of the fibers of two fiber bundles which appear in his geometric definition of the product, namely in the definition of the induction functor. In his theory, the Euler form appears in the definition of coproduct, namely in the definition of the restriction functor.

Now, Lusztig's version of the *Hall algebra* associated to Γ is the $\mathbb{C}(v)$-algebra with basis $\{u_\psi \mid \psi \text{ is a multisegment}\}$ and product is given by

$$u_{\varphi_1} u_{\varphi_2} = v^{m(\underline{\dim}\, T,\, \underline{\dim}\, W)} \sum_\psi F^\psi_{\varphi_1,\varphi_2}(v^{-2}) u_\psi.$$

Note that $[i;1)$ is the multisegment which labels the simple module S_i, for $i \in \mathbb{Z}/e\mathbb{Z}$. Then the $\mathbb{C}(v)$-subalgebra generated by these $u_{[i;1)}$ is called the *composition algebra*, and we may and do identify it with U_v^- by $u_{[i;1)} \mapsto f_i$. For the proof, see [23, Theorem 1.20].

Definition 2.6. For each multisegment ψ, we define $E_\psi = v^{\dim \mathcal{O}_\psi} u_\psi$. The set $\{E_\psi \mid \psi \text{ is a multisegment.}\}$ is called the PBW basis of the Hall algebra.

Example 2.7. Let $e = 3$. Then we have

$$f_1 f_2 = E_{\{[1;2)\}} + vE_{\{[1;1),[2;1)\}}, \quad f_2 f_1 = E_{\{[1;1),[2;1)\}}.$$

Similarly, we have

$$\begin{aligned}
f_0 f_1 f_2 =& E_{\{[0;3)\}} + vE_{\{[1;2),[0;1)\}} + vE_{\{[0;2),[2;1)\}} + v^2 E_{\{[0;1),[1;1),[2;1)\}}, \\
f_2 f_1 f_0 =& E_{\{[2;2),[1;1)\}} + vE_{\{[0;1),[1;1),[2;1)\}}, \\
f_0 f_2 f_1 =& E_{\{[0;2),[2;1)\}} + vE_{\{[0;1),[1;1),[2;1)\}}, \\
f_1 f_0 f_2 =& E_{\{[1;2),[0;1)\}} + vE_{\{[0;1),[1;1),[2;1)\}}.
\end{aligned}$$

Note that $E_{\{[0;1),[1;1),[2;1)\}}$ does not appear with coefficient 1. This is general phenomenon and we need aperiodicity to describe it.

Definition 2.8. A multisegment ψ is *aperiodic* if, for every $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that the segment of length l and head i does not appear in ψ. Equivalently, a multisegment ψ is aperiodic if, for each $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that the segment of length l and tail i does not appear in ψ.

The notion of aperiodicity and the following theorem are due to Lusztig. See [19, 15.3] and [20, Theorem 5.9].

Theorem 2.9. *For each $b \in B(\infty)$, the canonical basis element $G_v(b)$ has the form*

$$G_v(b) = E_\psi + \sum_{\psi' \neq \psi} c_{\psi,\psi'}(v) E_{\psi'},$$

for a unique aperiodic multisegment ψ, such that $c_{\psi,\psi'}(v) \in \mathbb{C}(v)$ is regular at $v = 0$ and $c_{\psi,\psi'}(0) = 0$.

Hence, we may label elements of $B(\infty)$ by aperiodic multisegments. We identify $B(\infty)$ with the set of aperiodic multisegments. Then, we denote the canonical basis by $G_v(\psi)$, for multisegments ψ, hereafter.

Leclerc, Thibon and Vasserot described the crystal structure on the set of aperiodic multisegments $B(\infty)$ in [18, Theorem 4.1], by using a result by Reineke.

Let ψ be a multisegment. Let $\psi_{\geq l}$ be the multisegment obtained from ψ by deleting multisegments of length less than l, for $l \in \mathbb{Z}_{>0}$. Let $m_{[i;l)}$ be the multiplicity of $[i;l)$ in ψ. Then, for $i \in \mathbb{Z}/e\mathbb{Z}$, we consider

$$S_{l,i} = \sum_{k \geq l} (m_{[i+1;k)} - m_{[i;k)}),$$

that is, the number of addable i-nodes of $\psi_{\geq l}$ minus the number of removable i-nodes of $\psi_{\geq l}$. Let $\ell_0 < \ell_1 < \cdots$ be those l that attain $\min_{l>0} S_{l,i}$. The following is the description of the crystal structure given by Leclerc, Thibon and Vasserot.

Theorem 2.10. *Let ψ be a multisegment, $i \in \mathbb{Z}/e\mathbb{Z}$ and let ℓ_0 be as above. Then, $\tilde{f}_i\psi = \psi_{\ell_0,i}$, where $\psi_{\ell_0,i}$ is obtained from ψ by adding $[i;1)$ if $\ell_0 = 1$, and by replacing $[i+1;\ell_0-1)$ with $[i;\ell_0)$ if $\ell_0 > 1$.*

2.3. **An anti-automorphism of U_v^-.** As the identification of the affine Hecke algebra with the convolution algebra $K^{G_n\times\mathbb{C}^\times}(Z_n)$, which will be explained in the next section, is not canonical, we go back and forth between two identifications. For this reason, we need another labelling by aperiodic multisegments.

Let $V = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}} V_i$ be a graded vector space as before, and define its dual graded vector space by $V^* = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}} V_i^*$ where $V_i^* = \mathrm{Hom}_{\mathbb{C}}(V_{-i},\mathbb{C})$. Then, by sending $X \in E_V$ to its transpose, we have a linear isomorphism

$$\rho : E_V \simeq E_{V^*} = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(V_i^*, V_{i+1}^*).$$

Using the standard basis of E_V and its dual basis in E_{V^*}, we identify the underlying spaces E_V and E_{V^*}. Note that the G_V-action on this E_V is the conjugation by the transpose inverse of $g \in G_V$, while the G_V-action on the original E_V is the conjugation by $g \in G_V$. Then, ρ is an isomorphism of two G_V-varieties E_V so that the G_V-orbit $\mathcal{O}_\psi$ in the original E_V corresponds to the G_V-orbit $\mathcal{O}_{\rho(\psi)}$ in the new E_V, where $\rho(\psi)$ is defined by $\rho([i;l)) = (l;-i]$. Thus, we have a linear isomorphism of the Hall algebras on both sides, which we also denote by ρ, such that

$$\rho(E_\psi) = E_{\rho(\psi)} \text{ and } \rho(G_v(\psi)) = G_v(\rho(\psi)) \text{ if } \psi \text{ is aperiodic.}$$

That is, this gives a relabelling of the PBW basis and the canonical basis. However, if we take the algebra structure into account, ρ induces the anti-automorphism of U_v^- given by $f_i \mapsto f_{-i}$, which is clear from the definition of the multiplication of the Hall algebra. In particular, the crystal structure on the set of aperiodic multisegments is changed in this new labelling, and the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ correspond to the Kashiwara operators $\tilde{e}_{-i}$ and $\tilde{f}_{-i}$ in this new crystal structure. In the new crystal structure, we change the definition of addable and removable i-nodes as follows.

Definition 2.11. We say that $(l;i]$ has a *removable i-node* and $(l;i-1]$ has an *addable i-node.*

We consider $S_{l,i} = \sum_{k\geq l}(m_{(k;i-1]} - m_{(k;i]})$, that is, the number of addable i-nodes of $\psi_{\geq l}$ minus the number of removable i-nodes of $\psi_{\geq l}$ in the new definition of removable and addable i-nodes. Let $\ell_0 < \ell_1 < \cdots$ be those l that attain $\min_{l>0} S_{l,i}$. Then, the crystal structure in the new labelling is given as follows. In fact, this version is stated in [18].

Theorem 2.12. *Let ψ be a multisegment, $i \in \mathbb{Z}/e\mathbb{Z}$ and let ℓ_0 be as above. Then, $\tilde{f}_i\psi = \psi_{\ell_0,i}$, where $\psi_{\ell_0,i}$ is obtained from ψ by adding $(1;i]$ if $\ell_0 = 1$, and by replacing $(\ell_0 - 1; i-1]$ with $(\ell_0; i]$ if $\ell_0 > 1$.*

To compute $\tilde{e}_i\psi$, for a multisegment ψ, we consider the same $S_{l,i}$. If $\min_{l>0} S_{l,i} = 0$, then $\tilde{e}_i\psi = 0$. Otherwise, let ℓ_0 be the maximal l that attains $\min_{l>0} S_{l,i}$. Then, $\tilde{e}_i\psi$ is obtained from ψ by replacing $(\ell_0; i]$ with $(\ell_0 - 1; i-1]$.

We use the crystal structure on the set of aperiodic multisegments in Theorem 2.10 when we choose the identification of $R(G_n \times \mathbb{C}^\times)$-algebras $H_n \simeq K^{G_n\times\mathbb{C}^\times}(Z_n)$ following Lusztig [22], while we use that in Theorem 2.12 when we choose the identification $H_n \simeq K^{G_n\times\mathbb{C}^\times}(Z_n)$ following Ginzburg [6]. We note that the second crystal structure is the star crystal structure of the first.

3. Affine Hecke algebras

Let H_n be the extended affine Hecke algebra associated with G_n. It is the $\mathbb{C}[q^{\pm 1}]$-algebra generated by T_i, for $1 \le i < n$, and $X_i^{\pm 1}$, for $1 \le i \le n$, subject to the relations

$$(T_i - q)(T_i + 1) = 0, \quad q^{-1}T_iX_iT_i = X_{i+1}, \quad \text{etc.}$$

In this section, we recall the geometric realization of affine Hecke algebras by Lusztig and Ginzburg, and of specialized affine Hecke algebras by Ginzburg.

3.1. **Varieties.** Let $G_n = GL_n(\mathbb{C})$ as before, and B_n the Borel subgroup of upper triangular matrices. We denote the unipotent radical of B_n by U_n, and the maximal torus of diagonal matrices by T_n. Write $\mathbb{C}^n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ and let $\mathcal{F}\ell_n$ be the flag variety, which consists of increasing subspaces $F = (F_i)_{0\le i\le n}$ in $\mathbb{C}^n$ such that $\dim F_i = i$, for all i. We consider the diagonal G_n-action on $\mathcal{F}\ell_n \times \mathcal{F}\ell_n$. Then, G_n-orbits in $\mathcal{F}\ell_n \times \mathcal{F}\ell_n$ are in bijection with B_n-orbits in $\mathcal{F}\ell_n$ and if we denote

$$\{(F, F') \in \mathcal{F}\ell_n \times \mathcal{F}\ell_n \mid \dim(F_i \cap F'_j) = \sharp\{k \mid 1 \le k \le i,\ 1 \le w(k) \le j\}\}$$

by $O_n(w)$, for $w \in \mathfrak{S}_n$, they give a complete set of G_n-orbits, and a pair of flags (F, F') belongs to $O_n(w)$ if and only if

$$\dim \frac{F_i \cap F'_j}{F_{i-1} \cap F'_j + F_i \cap F'_{j-1}} = \begin{cases} 1 & (j = w(i)) \\ 0 & (\text{otherwise}) \end{cases}.$$

We denote by $\mathcal{N}_n$ the set of nilpotent elements in $Mat_n(\mathbb{C})$ and write

$$Y_n = \{(X, F) \in \mathcal{N}_n \times \mathcal{F}\ell_n \mid XF_i \subseteq F_{i-1}\} \simeq T^*\mathcal{F}\ell_n.$$

Then the Steinberg variety is defined by

$$\begin{aligned} Z_n &= Y_n \times_{\mathcal{N}_n} Y_n \\ &= \{(X, F, F') \in \mathcal{N}_n \times \mathcal{F}\ell_n \times \mathcal{F}\ell_n \mid XF_i \subseteq F_{i-1}, XF'_i \subseteq F'_{i-1}\}. \end{aligned}$$

Z_n is a $G_n \times \mathbb{C}^\times$-variety by the action

$$(g,c)(X,F,F') = (c^{-1}gXg^{-1}, gF, gF'),$$

for $(g,c) \in G_n \times \mathbb{C}^\times$ and $(X,F,F') \in Z_n$.

We consider the complexified K-group of the abelian category of $G_n \times \mathbb{C}^\times$-equivariant coherent sheaves on Z_n. Using the closed embedding $Z_n \subseteq Y_n \times Y_n$, we have the convolution algebra $K^{G_n\times\mathbb{C}^\times}(Z_n)$. Z_n has a partition $Z_n = \sqcup_{w\in\mathfrak{S}_n} Z_n(w)$, where

$$Z_n(w) = \{(X,F,F') \in Z_n \mid (F,F') \in O_n(w)\}.$$

We have $\dim Z_n(w) = n(n-1)$ and $Z_n(w)$ is a $(\frac{n(n-1)}{2} - \ell(w))$-dimensional vector bundle over $O_n(w)$. Then, $\{\overline{Z_n(w)}\}_{w\in\mathfrak{S}_n}$ is the set of the irreducible components of Z_n. Define

$$Z_{n-1,n} = \{(X,F,F') \in Z_n \mid F_{n-1} = F'_{n-1}\}.$$

The condition $F_{n-1} = F'_{n-1}$ is equivalent to $(F,F') \in \sqcup_{w\in\mathfrak{S}_{n-1}} O_n(w)$, so that we have $Z_{n-1,n} = \sqcup_{w\in\mathfrak{S}_{n-1}} Z_n(w)$.

Similarly, $(F,F') \in O_n(e) \sqcup O_n(s_i) = \overline{O_n(s_i)}$ if and only if $F_j = F'_j$, for all $j \neq i$, and

$$\overline{Z_n(s_i)} = \{(X,F,F') \in Z_n \mid F_j = F'_j, \text{ for all } j \neq i,\ XF_{i+1} \subseteq F_{i-1}\}.$$

The pushforward of $\mathcal{O}_{\overline{Z_n(s_i)}}$ with respect to the closed embedding $\overline{Z_n(s_i)} \subseteq Z_n$ is also denoted by $\mathcal{O}_{\overline{Z_n(s_i)}}$ by abuse of notation. We denote

$$b_i = [\mathcal{O}_{\overline{Z_n(s_i)}}] \in K^{G_n\times\mathbb{C}^\times}(Z_n).$$

Let $Q_{i,i+1}$ be the parabolic subgroup of G_n which corresponds to s_i, $\mathfrak{n}_{i,i+1}$ the nilradical of its Lie algebra. Then

$$\overline{Z_n(s_i)} = (G_n \times \mathbb{C}^\times) \times_{Q_{i,i+1}\times\mathbb{C}^\times} (\mathfrak{n}_{i,i+1} \times \mathbb{P}^1 \times \mathbb{P}^1)$$

is a vector bundle over $\overline{O_n(s_i)} = (G_n \times \mathbb{C}^\times) \times_{Q_{i,i+1}\times\mathbb{C}^\times} (\mathbb{P}^1 \times \mathbb{P}^1)$. Then we define as follows.

Definition 3.1. The line bundle $\mathcal{L}_i$ on $\overline{Z_n(s_i)}$ is the pullback of

$$(G_n \times \mathbb{C}^\times) \times_{Q_{i,i+1}\times\mathbb{C}^\times} (\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$$

on $\overline{O_n(s_i)}$.

For $\lambda \in \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n = \mathrm{Hom}(T_n, \mathbb{C}^\times)$, let $\mathbb{C}_\lambda$ be the $B_n \times \mathbb{C}^\times$-module associated with λ and define the associated line bundle L_λ on $\mathcal{F}\ell_n$ by

$$L_\lambda = (G_n \times \mathbb{C}^\times) \times_{B_n\times\mathbb{C}^\times} \mathbb{C}_\lambda.$$

When we consider λ as a character of T_n, we denote it by e^λ. Then, we identify $K^{G_n\times\mathbb{C}^\times}(\mathcal{F}\ell_n) = R(T_n \times \mathbb{C}^\times)$ via $L_\lambda \mapsto e^\lambda$ as usual.

Let us denote $\pi_n : Y_n \to \mathcal{F}\ell_n$ and $\delta_n : Z_n(e) \subseteq Z_n$. We consider the diagram

$$\mathcal{F}\ell_n \xleftarrow{\pi_n} Y_n \simeq Z_n(e) \xrightarrow{\delta_n} Z_n$$

and we denote

$$\theta_\lambda = [\delta_{n*}\pi_n^* L_{-\lambda}] \in K^{G_n\times\mathbb{C}^\times}(Z_n).$$

Definition 3.2. We define $T_i = [\mathcal{L}_i] + q$, for $1 \le i < n$, and $X_i = \theta_{\epsilon_i}$, for $1 \le i \le n$.

We have $\theta_\lambda = \prod_{i=1}^n X_i^{\lambda_i}$, for $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i$. Using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\Delta\mathbb{P}^1} \to 0$$

where $\Delta\mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal, we know that $[\mathcal{L}_i] = b_i - (1 - q\theta_{\alpha_i})$.

Then, T_i, for $1 \le i < n$, and $X_i^{\pm1}$, for $1 \le i \le n$, satisfy the defining relations of H_n. In particular, we have the Bernstein relation

$$T_i\theta_\lambda = \theta_{s_i\lambda}T_i + (1-q)\frac{\theta_\lambda - \theta_{s_i\lambda}}{\theta_{-\alpha_i} - 1},$$

where $\alpha_i = -\epsilon_i + \epsilon_{i+1}$. This follows from the next theorem. The theorem was found by Lusztig and the action of T_i is called the Demazure-Lusztig operator.

Theorem 3.3. *Through the Thom isomorphism, we identify $K^{G_n\times\mathbb{C}^\times}(Y_n)$ with*

$$K^{G_n\times\mathbb{C}^\times}(\mathcal{F}\ell_n) = R(T_n \times \mathbb{C}^\times).$$

Then the convolution action of $K^{G_n\times\mathbb{C}^\times}(Z_n)$ on $K^{G_n\times\mathbb{C}^\times}(Y_n)$ is given by

$$T_i f = \frac{f - s_i f}{e^{\alpha_i} - 1} - q\frac{f - e^{\alpha_i}s_i f}{e^{\alpha_i} - 1}, \quad X_i f = e^{-\epsilon_i} f.$$

It is well-known that this is a faithful representation of H_n. Note that we have chosen the isomorphism $H_n \simeq K^{G_n\times\mathbb{C}^\times}(Z_n)$ to have the same formulas as [6, Theorem 7.2.16, Proposition 7.6.38]. When we follow [22], we define

$$\theta_\lambda = [\delta_{n*}\pi_n^* L_\lambda] \text{ and } T_i = -[\mathcal{L}_i] - 1.$$

Then, the formulas for the convolution action on $R(T_n \times \mathbb{C}^\times)$ change to those in [22, p.335]. The two identifications of $H_n \simeq K^{G_n\times\mathbb{C}^\times}(Z_n)$ are related by the involution σ defined by

$$T_i \mapsto -qT_i^{-1}, \quad X_i \mapsto X_i^{-1}.$$

In the rest of this section, we follow the identification in [22].

The center $Z(H_n)$ of H_n is the $\mathbb{C}[q^{\pm1}]$-subalgebra consisting of all the symmetric Laurent polynomials in $X_1, \ldots, X_n$. Thus, we identify $Z(H_n)$ with $R(G_n \times \mathbb{C}^\times)$. We also identify $\mathbb{C}[q^{\pm1}][X_1^{\pm1}, \ldots, X_n^{\pm1}]$ with $R(T_n \times \mathbb{C}^\times)$.

Let $K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n})$ be the convolution algebra with respect to the embedding $Z_{n-1,n} \subseteq Y_n \times Y_n$. Let

- $H_{n-1,n}$ be the parabolic subalgebra $H_{n-1} \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm1}]$ of H_n, and
- $\iota_n : Z_{n-1,n} \subseteq Z_n$ be the inclusion map.

We attribute the next theorem to Ginzburg [6] and Lusztig [22]. In [16], it was stated as an isomorphism of bimodules.

Theorem 3.4.

(1) *We have an isomorphism of* $R(G_n\times\mathbb{C}^\times)$*-algebras* $H_n \simeq K^{G_n\times\mathbb{C}^\times}(Z_n)$ *by the above choice of* T_i *and* X_i *in* $K^{G_n\times\mathbb{C}^\times}(Z_n)$*.*

(2) *The inclusion map* ι_n *induces the following commutative diagram of* $Z(H_n)$*-algebras.*

$$\begin{array}{rcl} \iota_{n*}: K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n}) & \to & K^{G_n\times\mathbb{C}^\times}(Z_n) \\ \downarrow & & \downarrow \\ H_{n-1,n} & \subseteq & H_n \end{array}$$

where the vertical arrows are isomorphisms.

It is also clear that the inclusion map $Y_n \simeq Z_n(e) \hookrightarrow Z_{n-1,n}$ induces

$$K^{G_n\times\mathbb{C}^\times}(Y_n) \to K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n})$$

and it is identified with $R(T_n\times\mathbb{C}^\times) \hookrightarrow H_{n-1,n}$.

3.2. **The embedding of H_{n-1} into H_n.** Let

$$Y_{n-1,n} = T^*\mathcal{F}\ell_n|_{\mathcal{F}\ell_{n-1}},$$

where we identify $\mathcal{F}\ell_{n-1} = \{F\in\mathcal{F}\ell_n \mid F_{n-1} = \mathbb{C}^{n-1}\}$, and let

$$\mathcal{N}_{n-1,n} = \{X\in\mathcal{N}_n \mid X\mathbb{C}^{n-1}\subseteq\mathbb{C}^{n-1}\}.$$

Then we define

$$\begin{aligned} Z'_{n-1,n} &= Y_{n-1,n}\times_{\mathcal{N}_{n-1,n}} Y_{n-1,n} \\ &= \{(X,F,F')\in Z_n \mid F_{n-1} = F'_{n-1} = \mathbb{C}^{n-1}\}. \end{aligned}$$

Let $P_{n-1,n}$ be the maximal parabolic subgroup of G_n that stabilizes $\mathbb{C}^{n-1}$. The Levi part $L_{n-1,n}\times\mathbb{C}^\times$ of $P_{n-1,n}\times\mathbb{C}^\times$ is $(G_{n-1}\times\mathbb{C}^\times)\times\mathbb{C}^\times$, which acts on Z_{n-1} by letting the middle component act trivially. We denote the unipotent radical of $P_{n-1,n}$ by $U_{n-1,n}$. It is also the unipotent radical of $P_{n-1,n}\times\mathbb{C}^\times$. Explicitly,

$$L_{n-1,n} = \begin{pmatrix} G_{n-1} & 0 \\ 0 & \mathbb{C}^\times \end{pmatrix}, \quad U_{n-1,n} = \left\{\begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix}\right\}.$$

We consider the following diagram.

$$Z'_{n-1,n} \xleftarrow{\mu_{n-1,n}} (G_n\times\mathbb{C}^\times)\times Z'_{n-1,n} \xrightarrow{\nu_{n-1,n}} (G_n\times\mathbb{C}^\times)\times_{P_{n-1,n}\times\mathbb{C}^\times} Z'_{n-1,n} = Z_{n-1,n}.$$

Then we have the restriction map

$$\mathrm{Res}^{G_n\times\mathbb{C}^\times}_{P_{n-1,n}\times\mathbb{C}^\times} : K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n}) \simeq K^{P_{n-1,n}\times\mathbb{C}^\times}(Z'_{n-1,n}).$$

$Z'_{n-1,n}$ is a $L_{n-1,n}\times\mathbb{C}^\times$-equivariant vector bundle of rank $n-1$ over Z_{n-1} and we write $\kappa_{n-1,n} : Z'_{n-1,n}\to Z_{n-1}$. Then $\kappa^*_{n-1,n}$ gives the Thom isomorphism

$$K^{L_{n-1,n}\times\mathbb{C}^\times}(Z'_{n-1,n}) \overset{\sim}{\leftarrow} K^{L_{n-1,n}\times\mathbb{C}^\times}(Z_{n-1}).$$

Noting that

$$K^{P_{n-1,n}\times\mathbb{C}^\times}(Z'_{n-1,n}) \simeq K^{L_{n-1,n}\times\mathbb{C}^\times}(Z'_{n-1,n})$$

by the forgetful map, and

$$K^{L_{n-1,n}\times\mathbb{C}^\times}(Z_{n-1}) \simeq K^{G_{n-1}\times\mathbb{C}^\times}(Z_{n-1}) \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm 1}],$$

we have

$$K^{P_{n-1,n}\times\mathbb{C}^\times}(Z'_{n-1,n}) \simeq K^{G_{n-1}\times\mathbb{C}^\times}(Z_{n-1}) \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm 1}].$$

Now, the following holds.

Proposition 3.5. *We have the following isomorphism of $R(L_{n-1,n}\times\mathbb{C}^\times)$-algebras*

$$K^{G_n\times\mathbb{C}^\times}(Z_n) \supseteq K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n}) \simeq K^{G_{n-1}\times\mathbb{C}^\times}(Z_{n-1}) \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm 1}],$$

which gets identified with $H_n \supseteq H_{n-1,n} = H_{n-1} \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm 1}]$.

Proof. We only have to show that $b_i \mapsto b_i$, for $1 \le i < n-1$, and $\theta_\lambda \mapsto \theta_\lambda$. Define

$$\overline{Z'_{n-1,n}(s_i)} = \{(X,F,F') \in Z'_{n-1,n} \mid F_j = F'_j \text{ for all } j \neq i,\ XF_{i+1} \subseteq F_{i-1}\}.$$

Then $\nu^{-1}_{n-1,n}(\overline{Z_n(s_i)}) = (G_n\times\mathbb{C}^\times)\times\overline{Z'_{n-1,n}(s_i)}$ and we have

$$\nu^*_{n-1,n}\mathcal{O}_{\overline{Z_n(s_i)}} = \mu^*_{n-1,n}\mathcal{O}_{\overline{Z'_{n-1,n}(s_i)}},\quad \mathcal{O}_{\overline{Z'_{n-1,n}(s_i)}} = \kappa^*_{n-1,n}\mathcal{O}_{\overline{Z_{n-1}(s_i)}}.$$

Hence, $b_i \mapsto b_i$, for $1 \le i < n-1$.

Let $Z'_{n-1,n}(e) = \{(X,F,F') \in Z'_{n-1,n} \mid F = F'\}$ and consider the diagram

$$\begin{array}{ccc} & & \nu^{-1}_{n-1,n}(Z_n(e)) = (G_n\times\mathbb{C}^\times)\times Z'_{n-1,n}(e) \\ & \swarrow & \downarrow \nu_{n-1,n} \\ Z'_{n-1,n}(e) \simeq Y_{n-1,n} & \subseteq & Y_n \simeq Z_n(e) \\ \pi_{n-1,n}\downarrow & & \downarrow \pi_n \\ \mathcal{F}\ell_{n-1} & \subseteq & \mathcal{F}\ell_n \end{array}$$

Then $\nu^*_{n-1,n}\pi^*_n L_\lambda = \mu^*_{n-1,n}\pi^*_{n-1,n}L_\lambda|_{\mathcal{F}\ell_{n-1}}$ and

$$L_\lambda|_{\mathcal{F}\ell_{n-1}} = (P_{n-1,n}\times\mathbb{C}^\times)\times_{B_n\times\mathbb{C}^\times}\mathbb{C}_\lambda.$$

But the diagram

$$\begin{array}{ccc} Z'_{n-1,n}(e) = \kappa^{-1}_{n-1,n}(Z_{n-1}(e)) \simeq Y_{n-1,n} & \xrightarrow{\kappa_{n-1,n}} & Y_{n-1} \simeq Z_{n-1}(e) \\ & \searrow & \downarrow \pi_{n-1} \\ & & \mathcal{F}\ell_{n-1} \end{array}$$

shows

$$\pi^*_{n-1,n}L_\lambda|_{\mathcal{F}\ell_{n-1}} = \kappa^*_{n-1,n}\left(\pi^*_{n-1}((P_{n-1,n}\times\mathbb{C}^\times)\times_{B_n\times\mathbb{C}^\times}\mathbb{C}_\lambda)\right).$$

Hence, $\theta_\lambda \mapsto \theta_\lambda$, for $\lambda \in \mathrm{Hom}(T_n,\mathbb{C}^\times)$.

As the generators b_i and θ_λ correspond correctly, it is an isomorphism of $R(L_{n-1,n}\times\mathbb{C}^\times)$-algebras, which is identified with $H_{n-1}\otimes\mathbb{C}[X_n^{\pm 1}] \hookrightarrow H_n$. $\square$

3.3. **Specialized Hecke algebras.** Let $\zeta \in \mathbb{C}$ be a primitive e^{th} root of unity, for $e \geq 2$. We fix a diagonal matrix $s = \mathrm{diag}(\zeta^{s_1}, \ldots, \zeta^{s_n})$, and set $a = (s, \zeta) \in G_n \times \mathbb{C}^\times$. We denote by A the smallest closed algebraic subgroup of $G_n \times \mathbb{C}^\times$ that contains a, namely the cyclic group $\langle a \rangle$ of order e in our case. Note that A is contained in $(G_{n-1} \times \mathbb{C}^\times) \times \mathbb{C}^\times$.

Definition 3.6. We denote the A-fixed points of M by M^a, for $M = Z_n$, $Z_{n-1,n}$, $Y_n = T^*\mathcal{F}\ell_n$, $\mathcal{F}\ell_n$ etc.

Let $\mathbb{C}_a$ be the $R(T_n \times \mathbb{C}^\times)$-module defined by $X_i \mapsto \zeta^{s_i}$, for $1 \leq i \leq n$, and $q \mapsto \zeta$. $\mathbb{C}_a|_{R(G_n \times \mathbb{C}^\times)}$ defines a central character $Z(H_n) = R(G_n \times \mathbb{C}^\times) \to \mathbb{C}$. Then we write $\mathbb{C}_a \otimes_{Z(H_n)} -$, for the specialization of the center with respect to the central character. We define $f_a \in \mathbb{C}[X_n]$ by

$$f_a(X_n) = (X_n - \zeta^{s_1}) \cdots (X_n - \zeta^{s_n}).$$

Definition 3.7. The $\mathbb{C}$-algebra $H_n^a = \mathbb{C}_a \otimes_{Z(H_n)} H_n$ is called the *specialized Hecke algebra* of rank n at a. The specialized algebra $\mathbb{C}_a \otimes_{Z(H_n)} H_{n-1,n}$ of the parabolic subalgebra $H_{n-1,n}$ is denoted $H_{n-1,n}^a$.

Lemma 3.8. *Let a_k be the k-th elementary symmetric function in $X_1, \ldots, X_n$ evaluated at $X_1 = \zeta^{s_1}, \ldots, X_n = \zeta^{s_n}$. Then, $\mathbb{C}[X_n^{\pm 1}]/(f_a)$ is a $Z(H_{n-1})$-algebra via*

$$e_k \mapsto a_k - a_{k-1}X_n + \cdots + (-1)^k X_n^k,$$

where e_k is the k-th elementary symmetric function in $X_1, \ldots, X_{n-1}$, and

$$H_{n-1,n}^a = H_{n-1} \otimes_{Z(H_{n-1})} \mathbb{C}[X_n^{\pm 1}]/(f_a).$$

Proof. As $e_k + X_n e_{k-1}$ is the k-th elementary symmetric function in $X_1, \ldots, X_n$, the surjective map

$$H_{n-1,n} = H_{n-1} \otimes_{\mathbb{C}} \mathbb{C}[X_n^{\pm 1}] \to H_{n-1} \otimes_{Z(H_{n-1})} \mathbb{C}[X_n^{\pm 1}]/(f_a)$$

factors through $H_{n-1,n}^a$. On the other hand, both have the same dimension $n!(n-1)!$. Hence the result. $\square$

As $H_n \simeq K^{G_n \times \mathbb{C}^\times}(Z_n)$ and $H_{n-1,n} \simeq K^{G_n \times \mathbb{C}^\times}(Z_{n-1,n})$ as $R(G_n \times \mathbb{C}^\times)$-algebras, we identify the following $\mathbb{C}$-algebras respectively.

$$H_n^a = \mathbb{C}_a \otimes_{R(G_n \times \mathbb{C}^\times)} K^{G_n \times \mathbb{C}^\times}(Z_n),$$
$$H_{n-1,n}^a = \mathbb{C}_a \otimes_{R(G_n \times \mathbb{C}^\times)} K^{G_n \times \mathbb{C}^\times}(Z_{n-1,n}).$$

By Proposition 3.5, geometric realization of Lemma 3.8 is given by

$$\begin{aligned}\mathbb{C}_a \otimes_{Z(H_n)} K^{G_n \times \mathbb{C}^\times}(Z_{n-1,n}) &\simeq \mathbb{C}_a \otimes_{Z(H_n)} \left(K^{G_{n-1} \times \mathbb{C}^\times}(Z_{n-1}) \otimes \mathbb{C}[X_n^{\pm 1}]\right) \\ &\simeq K^{G_{n-1} \times \mathbb{C}^\times}(Z_{n-1}) \otimes_{Z(H_{n-1})} \mathbb{C}[X_n^{\pm 1}]/(f_a).\end{aligned}$$

Let m_i be the multiplicity of ζ^i in $\{\zeta^{s_1}, \ldots, \zeta^{s_n}\}$. Then

$$\mathbb{C}[X_n^{\pm 1}]/(f_a) \simeq \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{C}[X_n^{\pm 1}]/((X_n - \zeta^i)^{m_i}).$$

Definition 3.9. We denote by p_i the identity of $\mathbb{C}[X_n^{\pm 1}]/((X_n-\zeta^i)^{m_i})$ which is viewed as an element of $H^a_{n-1,n}$. Thus, p_i are central idempotents of $H^a_{n-1,n}$ such that $\sum_{i\in\mathbb{Z}/e\mathbb{Z}} p_i = 1$ and $p_ip_j = p_jp_i = \delta_{ij}p_i$.

We have the decomposition of $\mathbb{C}$-algebras

$$H^a_{n-1,n} = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} p_iH^a_{n-1,n}p_i.$$

We fix $i\in\mathbb{Z}/e\mathbb{Z}$ and suppose that $(\nu_1,\dots,\nu_n)$ is a permutation of $(s_1,\dots,s_n)$ such that $\nu_n = i$. Then

$$(\mathrm{diag}(\zeta^{\nu_1},\dots,\zeta^{\nu_{n-1}}),\zeta)\in G_{n-1}\times\mathbb{C}^\times$$

defines a central character of H_{n-1} and we may define the specialized Hecke algebra with respect to the central character. We denote it by $H^{a;i}_{n-1}$. By Lemma 3.8, we have the surjective algebra homomorphism

$$p_iH^a_{n-1,n}p_i \longrightarrow H^{a;i}_{n-1},$$

because if we write $b_k = a_k - a_{k-1}\zeta^i + \cdots + (-1)^k\zeta^{ki}$, then

$$\begin{aligned}\left(\sum_{k=0}^{n-1}(-1)^k b_kT^k\right)(1-\zeta^{\nu_n}T) &= \sum_{k=0}^{n}(-1)^k(b_{k-1}\zeta^i+b_k)T^k\\ &= \sum_{k=0}^{n}(-1)^k a_kT^k\\ &= (1-\zeta^{s_1}T)\cdots(1-\zeta^{s_n}T)\end{aligned}$$

and $e_k\mapsto b_k$, for $1\le k\le n-1$, is the central character which defines $H^{a;i}_{n-1}$. Composing it with the projection $H^a_{n-1,n}$ to $p_iH^a_{n-1,n}p_i$, we have

$$H^a_{n-1,n}\longrightarrow H^{a;i}_{n-1},$$

which is nothing but the specialization map at $X_n = \zeta^i$. Its geometric realization is given by

$$\mathbb{C}_a\otimes_{Z(H_n)} K^{G_n\times\mathbb{C}^\times}(Z_{n-1,n})\longrightarrow \mathbb{C}_{a;i}\otimes_{Z(H_{n-1})} K^{G_{n-1}\times\mathbb{C}^\times}(Z_{n-1}).$$

Lemma 3.10. *Simple $p_iH^a_{n-1,n}p_i$-modules are obtained from simple $H^{a;i}_{n-1}$-modules through the algebra homomorphism $p_iH^a_{n-1,n}p_i\to H^{a;i}_{n-1}$.*

Proof. Let I_a be the two-sided ideal of $p_iH^a_{n-1,n}p_i$ generated by $X_n-\zeta^i$. Then I_a is nilpotent, so that I_a acts as zero on simple $p_iH^a_{n-1,n}p_i$-modules. As $H^{a;i}_{n-1} = p_iH^a_{n-1,n}p_i/I_a$ by Lemma 3.8, we have the result. □

Ginzburg's theory tells us how to realize the specialized Hecke algebra in sheaf theory. Definitions of the maps in (1) are necessary in the proof of (2), so that they will be given in the proof.

Theorem 3.11.

(1) *We may identify* $H_n^a = H_*^{BM}(Z_n^a, \mathbb{C})$ *by*

$$\mathbb{C}_a \otimes_{R(A)} K^{G_n \times \mathbb{C}^\times}(Z_n) \simeq \mathbb{C}_a \otimes_{R(A)} K^A(Z_n) \overset{res_n}{\simeq} K(Z_n^a) \overset{RR_n}{\simeq} H_*^{BM}(Z_n^a, \mathbb{C}).$$

We may identify $H_{n-1,n}^a = H_*^{BM}(Z_{n-1,n}^a, \mathbb{C})$ *in the same way.*

(2) *The following diagram of* $\mathbb{C}$*-algebras commutes.*

$$\begin{array}{ccc} H_*^{BM}(Z_{n-1,n}^a, \mathbb{C}) & \xrightarrow{\iota_{n*}} & H_*^{BM}(Z_n^a, \mathbb{C}) \\ \| & & \| \\ H_{n-1,n}^a & \hookrightarrow & H_n^a \end{array}$$

Similarly, $Y_n^a \to Z_{n-1,n}^a$ *induces the following commutative diagram of* $\mathbb{C}$*-algebras.*

$$\begin{array}{ccc} H_*^{BM}(Y_n^a, \mathbb{C}) & \longrightarrow & H_*^{BM}(Z_{n-1,n}^a, \mathbb{C}) \\ \| & & \| \\ \mathbb{C}_a \otimes_{Z(H_n)} R(T_n \times \mathbb{C}^\times) & \hookrightarrow & H_{n-1,n}^a \end{array}$$

Proof. (1) is well-known. See [4] or [6]. We check the commutativity in (2). Let $Y_n^a \times Y_n^a \subseteq Y_n \times Y_n$ be the closed embedding. Then, we have

$$\begin{aligned} i_n^A : \ & Z_n^a = Z_n \cap (Y_n^a \times Y_n^a) \hookrightarrow Z_n, \\ i_{n-1,n}^A : \ & Z_{n-1,n}^a = Z_{n-1,n} \cap (Y_n^a \times Y_n^a) \hookrightarrow Z_{n-1,n}. \end{aligned}$$

We define the pullback

$$\begin{aligned} (i_n^A)^* : \quad & K^A(Z_n) \longrightarrow K^A(Z_n^a), \\ (i_{n-1,n}^A)^* : \quad & K^A(Z_{n-1,n}) \longrightarrow K^A(Z_{n-1,n}^a), \end{aligned}$$

in terms of the embedding $Y_n^a \times Y_n^a \subseteq Y_n \times Y_n$. We have the linear A-action on each fiber of the normal bundle $T_{Y_n^a} Y_n$ and its decomposition into isotropic components leads to the decomposition of the normal bundle into the direct sum of vector bundles N_i, for $i \in \mathbb{Z}/e\mathbb{Z}$, over Y_n^a. We define λ_n by

$$\lambda_n = \bigotimes_{i \in \mathbb{Z}/e\mathbb{Z}} \left(\sum_{j \geq 0} (-\zeta^i)^j \wedge^j N_i^\vee \right) \in K(Y_n^a).$$

res_n for Z_n is defined by

$$res_n : K^A(Z_n) \xrightarrow{(i_n^A)^*} \mathbb{C}_a \otimes_{R(A)} K^A(Z_n^a) \simeq K(Z_n^a) \xrightarrow{1 \otimes \lambda_n^{-1}} K(Z_n^a),$$

and similarly for $Z_{n-1,n}$ and Y_n. Here, $1\otimes\lambda_n^{-1}\in K(Y_n^a\times Y_n^a)$ acts on $K(Z_n^a)$ by the multiplication. Then, the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}_a\otimes_{R(A)} K^A(Y_n) & \simeq & K(Y_n^a) \\ \downarrow & & \downarrow \\ \mathbb{C}_a\otimes_{R(A)} K^A(Z_{n-1,n}) & \simeq & K(Z_{n-1,n}^a) \\ \iota_{n*}\downarrow & & \downarrow\iota_{n*} \\ \mathbb{C}_a\otimes_{R(A)} K^A(Z_n) & \simeq & K(Z_n^a) \end{array}$$

follows from the statement below.

> Let $N\subseteq M$ be a closed embedding between smooth varieties, Z a closed subvariety of M. Let $Z'=Z\cap N$ and denote
>
> $$\begin{array}{ccc} N & \xrightarrow{\psi} & M \\ \iota'\uparrow & & \uparrow\iota \\ Z' & \xrightarrow{\psi'} & Z \end{array}$$
>
> We define ψ'^* with respect to these inclusions to smooth varieties. Then, $\psi^*\iota_*[\mathcal{F}]=\iota'_*\psi'^*[\mathcal{F}]$.

To see this, observe that both sides are essentially the same $[\psi_*\mathcal{O}_N\otimes^L_{\mathcal{O}_M}\iota_*\mathcal{F}]$ by the definition of ψ'^*.

Finally, recalling that RR_n is defined by

$$RR_n(\mathcal{F})=ch(\mathcal{F})(1\otimes td_{Y_n^a})\cap[Y_n^a\times Y_n^a],$$

we have the commutativity in (2). □

Theorem 3.11 (1) also allows us to identify $H_{n-1,n}^a\to H_{n-1}^{a;i}$ with

$$H_*^{BM}(Z_{n-1,n}^a,\mathbb{C})\longrightarrow H_*^{BM}(Z_{n-1}^{a;i},\mathbb{C}),$$

but we do not try to find the geometric realization of this homomorphism. Later, $p_iH_{n-1,n}^ap_i\to H_{n-1}^{a;i}$ will be identified with

$$\mathrm{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C})\longrightarrow \mathrm{Ext}^*_{D^b(\mathcal{N}_{n-1}^a)}(R\pi^{a;i}_{n-1!}\mathbb{C}),$$

where the homomorphism is defined by the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C}) & \longrightarrow & \mathrm{Ext}^*_{D^b(\mathcal{N}_{n-1}^a)}(R\pi^{a;i}_{n-1!}\mathbb{C}) \\ \| & & \| \\ p_iH_{n-1,n}^ap_i & \longrightarrow & H_{n-1}^a \end{array}$$

Then, we consider the induced isomorphism

$$p_iH_{n-1,n}^ap_i/\operatorname{Rad}(p_iH_{n-1,n}^ap_i)\simeq H_{n-1}^{a;i}/\operatorname{Rad}(H_{n-1}^{a;i})$$

in this identification. Lemma 4.11 shows that the isomorphism may be given geometrically and it suffices for our purpose.

Recall that we have identified $e^\lambda \in R(T_n \times \mathbb{C}^\times)$ with $\prod_{i=1}^n X_i^{\lambda_i} \in H_n$. Denote the product by X^λ. Then, in the above theorem, $1 \otimes X^\lambda$ is identified with

$$ch(\pi_n^* L_\lambda|_{Y_n^a}) td_{Y_n^a} ch(\lambda_n)^{-1} \cap [Y_n^a] \in H_*^{BM}(Y_n^a, \mathbb{C}).$$

In particular, the identity element of $H_*^{BM}(Y_n^a, \mathbb{C})$ is $td_{Y_n^a} ch(\lambda_n)^{-1} \cap [Y_n^a]$ and the multiplication by X^λ is the same as the cap product $ch(\pi_n^* L_\lambda|_{Y_n^a}) \cap -$.

4. Geometric proof of the modular branching rule

In this section, we give a geometric proof of the modular branching rule.

4.1. **The statement.** First we explain the precise statement which we are going to prove. In fact, we have two versions according to the choice of the identification $H_n = K^{G_n \times \mathbb{C}^\times}(Z_n)$.

Definition 4.1. For an H_n-module M, define the i-restriction

$$i\text{-}\operatorname{Res}(M) = \{m \in M \mid (X_n - \zeta^i)^N m = 0, \text{for large enough } N.\}.$$

Then, the statement of the modular branching rule is as follows. The modules L_ψ will be introduced in 4.4.

Theorem 4.2. *We identify H_n with $K^{G_n \times \mathbb{C}^\times}(Z_n)$ by $\theta_\lambda = [\delta_{n*} \pi_n^* L_\lambda]$ and $T_i = -[\mathcal{L}_i] - 1$. Then, for the simple H_n-module L_ψ labelled by an aperiodic multisegment ψ, we have*

$$\operatorname{Soc}(i\text{-}\operatorname{Res}(L_\psi)) = L_{\tilde{e}_i \psi},$$

where the crystal structure on the set of aperiodic multisegments is as in Theorem 2.10.

Let us consider the other identification of H_n with $K^{G_n \times \mathbb{C}^\times}(Z_n)$. Recall the involution σ defined by $T_i \mapsto -q T_i^{-1}$ and $X_i \mapsto X_i^{-1}$.

Definition 4.3. An H_n-module obtained from L_ψ by twisting the action by σ and relabelling aperiodic multisegments by ρ is denoted by

$$D_\psi = {}^\sigma L_{\rho(\psi)}.$$

Theorem 4.4. *We identify H_n with $K^{G_n \times \mathbb{C}^\times}(Z_n)$ by $\theta_\lambda = [\delta_{n*} \pi_n^* L_{-\lambda}]$ and $T_i = [\mathcal{L}_i] + q$. Then, for the simple H_n-module D_ψ labelled by an aperiodic multisegment ψ, we have*

$$\operatorname{Soc}(i\text{-}\operatorname{Res}(D_\psi)) = D_{\tilde{e}_i \psi},$$

where the crystal structure on the set of aperiodic multisegments is as in Theorem 2.12.

Theorem 4.4 follows from Theorem 4.2. In fact,

$$\begin{aligned}\operatorname{Soc}(i\text{-}\operatorname{Res}(D_\psi)) &\simeq \operatorname{Soc}({}^\sigma((-i)\text{-}\operatorname{Res}(L_{\rho(\psi)}))) \\ &\simeq {}^\sigma \operatorname{Soc}((-i)\text{-}\operatorname{Res}(L_{\rho(\psi)})) \simeq {}^\sigma L_{\tilde{e}_{-i}\rho(\psi)},\end{aligned}$$

where $\tilde{e}_{-i}$ is the Kashiwara operator with respect to the crystal structure in Theorem 4.2, so that it is isomorphic to ${}^{\sigma}L_{\rho(\tilde{e}_i\psi)} = D_{\tilde{e}_i\psi}$ where $\tilde{e}_i$ is the Kashiwara operator with respect to the crystal structure in Theorem 4.4.

In the rest of the section, we identify H_n with $K^{G_n\times\mathbb{C}^\times}(Z_n)$ as in Theorem 4.2 and prove the theorem.

4.2. **Localization and eigenvalues of X_n.** Suppose that $(X,F)\in Y_n^a$. Then, $sXs^{-1}=\zeta X$ and F is such that F_i is obtained from F_{i-1} by adding some eigenvector of s. We denote the eigenvalue of the eigenvector by ζ^{ν_i}, for $\nu_i\in\mathbb{Z}/e\mathbb{Z}$, and write $\nu=(\nu_1,\dots,\nu_n)$. We call ν the *flag type* of (X,F). Note that ν is a permutation of $(s_1,\dots,s_n)$. For $(X,F,F')\in Z_n^a=Y_n^a\times_{\mathcal{N}_n^a}Y_n^a$, we say that the *flag type* of (X,F,F') is (ν,ν') if (X,F) has flag type ν and (X,F') has flag type ν'.

Now, we look at the decomposition of Y_n^a and $Z_{n-1,n}^a$ into connected components. On each component, the flag type is constant.

Definition 4.5. Let $p_iY_n^a$ be the disjoint union of connected components of Y_n^a whose flag type ν satisfies $\nu_n=i$.

Similarly, we let $p_iZ_{n-1,n}^ap_i$ be the disjoint union of connected components of $Z_{n-1,n}^a$ whose flag type (ν,ν') satisfies $\nu_n=\nu'_n=i$.

The following lemma uses our choice of the identification of H_n with $K^{G_n\times\mathbb{C}^\times}(Z_n)$ in this section.

Lemma 4.6. *Under the identification $H_*^{BM}(Z_{n-1,n}^a,\mathbb{C})=H_{n-1,n}^a$, we have*

$$H_*^{BM}(p_iZ_{n-1,n}^ap_i,\mathbb{C})=p_iH_{n-1,n}^ap_i.$$

Proof. Let $(Y_n^a)_\mu$ be the set of $(X,F)\in Y_n^a$ such that the flag type is μ. First we show that

$$H_*^{BM}(Y_n^a,\mathbb{C})p_i=\bigoplus_{\mu\text{ such that }\mu_n=i}H_*^{BM}((Y_n^a)_\mu,\mathbb{C}).$$

In fact, X_n acts on $\mathbb{C}_a\otimes_{R(A)}K^A(Y_n^a)$ by

$$\pi_n^*L_{\epsilon_n}|_{Y_n^a}\otimes -$$

by Theorem 3.11. Now, A acts on fiberwise over Y_n^a, and the fiber of $\pi_n^*L_{\epsilon_n}$ at (X,F) is $\mathbb{C}^n/F_{n-1}$. Thus, A acts as ζ^{μ_n} on the fiber when the flag type of (X,F) is μ. Then, X_n is $\zeta^{\mu_n}\pi_n^*L_{\epsilon_n}|_{Y_n^a}\in K((Y_n^a)_\mu)$, where $\pi_n^*L_{\epsilon_n}|_{Y_n^a}$ is a line bundle without A-action, and Theorem 3.11 implies that X_n acts on $H_*^{BM}((Y_n^a)_\mu,\mathbb{C})$ by the cap product of

$$\zeta^{\mu_n}ch(\pi_n^*L_{\epsilon_n}|_{Y_n^a})=\zeta^{\mu_n}+\text{higher degree terms}.$$

Hence, $X_n-\zeta^{\mu_n}$ acts nilpotently on $H_*^{BM}((Y_n^a)_\mu,\mathbb{C})$. We have proved the claim.

Let ${}_\nu(Z_{n-1,n}^a)_{\nu'}$ be the set of $(X,F,F')\in Z_{n-1,n}^a$ such that the flag type is (ν,ν'). By the definition of the convolution product, the product

$$H_*^{BM}((Y_n^a)_\mu,\mathbb{C})\cdot H_*^{BM}({}_\nu(Z_{n-1,n}^a)_{\nu'},\mathbb{C})$$

is nonzero only if $\mu = \nu$. Thus, $p_i H_*^{BM}({}_\nu(Z^a_{n-1,n})_{\nu'}, \mathbb{C}) = 0$ if $\nu_n \neq i$, and the left multiplication by p_i acts as the identity map on $H_*^{BM}({}_\nu(Z^a_{n-1,n})_{\nu'}, \mathbb{C})$ if $\nu_n = i$. Similar argument shows that $H_*^{BM}({}_\nu(Z^a_{n-1,n})_{\nu'}, \mathbb{C})p_i = 0$ if $\nu'_n \neq i$, and the right multiplication by p_i acts as the identity map on $H_*^{BM}({}_\nu(Z^a_{n-1,n})_{\nu'}, \mathbb{C})$ if $\nu'_n = i$. We have proved the result. $\square$

4.3. **A functorial algebra homomorphism.** Now we work in the derived categories of abelian categories of sheaves of $\mathbb{C}$-vector spaces. The following is proved in [6, Proposition 8.6.35].

Theorem 4.7. *Let M_1, M_2 and M_3 be connected smooth varieties, N a variety and let $\mu_i : M_i \to N$ be proper maps. Let $\mathcal{A}_i \in D^b(M_i)$ be a constructible complex, for $i = 1, 2, 3$. Define $Z_{ij} = M_i \times_N M_j$ and denote $\iota_{ij} : Z_{ij} \subseteq M_i \times M_j$ the inclusion map. Let $\mathcal{A}_{ij} = \iota_{ij}^!(\mathcal{A}_i^\vee \otimes \mathcal{A}_j)$. Then the following hold.*

(1) *Let $\mu_{ij} : Z_{ij} \to N$ be the projection map. Then*

$$R\mu_{ij*}\mathcal{A}_{ij} \simeq R\mathcal{H}om(R\mu_{i*}\mathcal{A}_i, R\mu_{j*}\mathcal{A}_j).$$

Thus, we have isomorphisms of $\mathbb{C}$-algebras

$$H^*(Z_{ij}, \mathcal{A}_{ij}) = H^*(N, R\mu_{ij*}\mathcal{A}_{ij}) \simeq \mathrm{Ext}^*_{D^b(N)}(R\mu_{i*}\mathcal{A}_i, R\mu_{j*}\mathcal{A}_j).$$

(2) *The convolution product*

$$H^*(Z_{ij}, \mathcal{A}_{ij}) \otimes H^*(Z_{jk}, \mathcal{A}_{jk}) \longrightarrow H^*(Z_{ik}, \mathcal{A}_{ik})$$

is identified with the Yoneda product

$$\mathrm{Ext}^*_{D^b(N)}(R\mu_{i*}\mathcal{A}_i, R\mu_{j*}\mathcal{A}_j) \otimes \mathrm{Ext}^*_{D^b(N)}(R\mu_{j*}\mathcal{A}_j, R\mu_{k*}\mathcal{A}_k) \longrightarrow \mathrm{Ext}^*_{D^b(N)}(R\mu_{i*}\mathcal{A}_i, R\mu_{k*}\mathcal{A}_k)$$

under the isomorphisms in (1).

We view elements of $\mathcal{N}_n^a$ as representations of the cyclic quiver of length e. Namely, we put $V_i = \{v \in \mathbb{C}^n \mid sv = \zeta^i v\}$ on the i^{th} node, for $i \in \mathbb{Z}/e\mathbb{Z}$, then $X \in \mathcal{N}_n^a$ defines $X : V_i \to V_{i+1}$, for $i \in \mathbb{Z}/e\mathbb{Z}$.

We fix $i \in \mathbb{Z}/e\mathbb{Z}$. Let $m+1 = \dim V_i$ and $\mathbb{P}^m$ the projective space consisting of m-dimensional subspaces of V_i. We have the following commutative diagram.

$$\begin{array}{ccc}
p_iY_n^a \underset{\mathcal{N}_n^a \times \mathbb{P}^m}{\times} p_iY_n^a = p_iZ^a_{n-1,n}p_i & \hookrightarrow & p_iZ_n^ap_i = p_iY_n^a \underset{\mathcal{N}_n^a}{\times} p_iY_n^a \\
\downarrow & & \downarrow \\
\mathcal{N}_n^a \times \mathbb{P}^m & \underset{\rho_n}{\to} & \mathcal{N}_n^a
\end{array}$$

where $\rho_n(X, U) = X$ and the left vertical map is given by $(X, F, F') \mapsto (X, F_{n-1})$.

Lemma 4.8. *Let $M \xrightarrow{f} X \xrightarrow{g} Y$ be proper maps and suppose that M is smooth. We consider the following diagram, in which all squares are cartesian.*

$$\begin{array}{ccccc}
M \times_X M & \xrightarrow{\tilde{\iota}} & M \times_Y M & \xrightarrow{\tilde{\Delta}} & M \times M \\
\pi \downarrow & & \downarrow \pi' & & \downarrow f^{\times 2} \\
X & \xrightarrow{\iota} & X \times_Y X & \xrightarrow{\Delta} & X \times X \\
& g \searrow & \downarrow \pi'' & \swarrow g^{\times 2} & \\
& & Y \xrightarrow{\overline{\Delta}} Y \times Y & &
\end{array}$$

Denote $\mathcal{A} = Rf_*\mathbb{C}$ and $\mathcal{B} = Rg_*\mathcal{A}$. Then the following hold.
(1) *We have the following isomorphisms of $\mathbb{C}$-algebras.*

$$H_*^{BM}(M \times_X M, \mathbb{C}) \simeq \mathrm{Ext}^*_{D^b(X)}(\mathcal{A}, \mathcal{A}), \quad H_*^{BM}(M \times_Y M, \mathbb{C}) \simeq \mathrm{Ext}^*_{D^b(Y)}(\mathcal{B}, \mathcal{B}).$$

(2) $\tilde{\iota}_* : H_*^{BM}(M \times_X M, \mathbb{C}) \to H_*^{BM}(M \times_Y M, \mathbb{C})$ *is identified with the functorial algebra homomorphism*

$$Rg_* : \mathrm{Ext}^*_{D^b(X)}(\mathcal{A}, \mathcal{A}) \longrightarrow \mathrm{Ext}^*_{D^b(Y)}(\mathcal{B}, \mathcal{B}).$$

Proof. (1) follows from Theorem 4.7. In fact, if we ignore degree shift then

$$\begin{aligned}
H_*^{BM}(M \times_X M, \mathbb{C}) &\simeq H^*(M \times_X M, \tilde{\iota}^! \tilde{\Delta}^! \mathbb{C}) \simeq H^*(X, R\pi_* \tilde{\iota}^! \tilde{\Delta}^! \mathbb{C}) \\
&\simeq H^*(X, \iota^! R\pi'_* \tilde{\Delta}^! \mathbb{C}) \simeq H^*(X, \iota^! \Delta^! Rf_*^{\times 2} \mathbb{C}).
\end{aligned}$$

As $\mathcal{A}^\vee = (Rf_*\mathbb{C})^\vee = Rf_!\,\mathbb{C}^\vee = \oplus Rf_*\mathbb{C}[2\dim M_i]$, where the summation is over connected components M_i of M, if we ignore degree shift then

$$R\mathcal{H}om_{D^b(X)}(\mathcal{A}, \mathcal{A}) = (\Delta \circ \iota)^!(\mathcal{A}^\vee \otimes \mathcal{A}) = (\Delta \circ \iota)^! Rf_*^{\times 2}\mathbb{C}.$$

Hence, $H_*^{BM}(M \times_X M, \mathbb{C}) \simeq \mathrm{Ext}^*_{D^b(X)}(\mathcal{A}, \mathcal{A})$ is proved. We can prove the other isomorphism similarly.

(2) If we ignore degree shift, the pushforward ι_* of Borel-Moore homology groups is given by

$$H^*(M \times M, R(\tilde{\Delta} \circ \tilde{\iota})_*(\tilde{\Delta} \circ \tilde{\iota})^!(\mathbb{C}^\vee \otimes \mathbb{C})) \longrightarrow H^*(M \times M, R\tilde{\Delta}_* \tilde{\Delta}^!(\mathbb{C}^\vee \otimes \mathbb{C})).$$

First we claim that it is identified with

$$\Gamma(X \times_Y X, R\iota_* \iota^! \Delta^!(\mathcal{A}^\vee \otimes \mathcal{A})) \longrightarrow \Gamma(X \times_Y X, \Delta^!(\mathcal{A}^\vee \otimes \mathcal{A})).$$

To see this, let $\mathcal{I}^\bullet$ be an injective resolution of $\tilde{\Delta}^!(\mathbb{C}^\vee \otimes \mathbb{C})$. Then, for the complex of sheaves $\Gamma_{M \times_X M}(\mathcal{I}^\bullet)$, which is defined by

$$U \mapsto \Gamma_{M \times_X M}(\mathcal{I}^\bullet)(U) = \{s^\bullet \in \mathcal{I}^\bullet(U) \mid \mathrm{supp}(s^i) \subseteq M \times_X M, \text{ for all } i.\},$$

for open subsets $U \subseteq M \times_Y M$, the ι_* in question is obtained by taking the cohomology of the following morphism of complexes of $\mathbb{C}$-vector spaces.

$$\Gamma(M \times_Y M, \Gamma_{M \times_X M}(\mathcal{I}^\bullet)) \longrightarrow \Gamma(M \times_Y M, \mathcal{I}^\bullet).$$

For open subsets $U \subseteq X \times_Y X$, we have

$$\begin{aligned}\Gamma_X(\pi'_*\mathcal{F})(U) &= \operatorname{Ker}\left(\pi'_*\mathcal{F}(U) \overset{restriction}{\longrightarrow} \pi'_*\mathcal{F}(U \setminus X)\right)\\ &= \operatorname{Ker}\left(\mathcal{F}(\pi'^{-1}(U)) \overset{restriction}{\longrightarrow} \mathcal{F}(\pi'^{-1}(U) \setminus M \times_X M)\right)\\ &= \Gamma_{M\times_X M}(\mathcal{F})(\pi'^{-1}(U)),\end{aligned}$$

for a sheaf $\mathcal{F}$ on $X \times_Y X$, so that the above morphism of complexes of $\mathbb{C}$-vector spaces is nothing but

$$\Gamma(X \times_Y X, \Gamma_X(\pi'_*\mathcal{I}^\bullet)) \longrightarrow \Gamma(X \times_Y X, \pi'_*\mathcal{I}^\bullet),$$

and it is identified with

$$\Gamma(X \times_Y X, R\iota_*\iota^! R\pi'_*\tilde{\Delta}^!(\mathbb{C}^\vee \otimes \mathbb{C})) \longrightarrow \Gamma(X \times_Y X, R\pi'_*\tilde{\Delta}^!(\mathbb{C}^\vee \otimes \mathbb{C})).$$

Now we apply the natural transformation $R\iota_*\iota^! \to \mathrm{Id}$ to the isomorphism

$$R\pi'_*\tilde{\Delta}^!(\mathbb{C}^\vee \otimes \mathbb{C}) \simeq \Delta^! Rf_*^{\times 2}(\mathbb{C}^\vee \otimes \mathbb{C}) \simeq \Delta^!((Rf_!\mathbb{C})^\vee \otimes Rf_*\mathbb{C}))$$

to obtain the claim.

Next let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{A}$. Then, our morphism of complexes of $\mathbb{C}$-vector spaces is

$$\Gamma(X \times_Y X, \Gamma_X(\mathcal{I}^{\bullet\vee} \otimes \mathcal{I}^\bullet)) \longrightarrow \Gamma(X \times_Y X, \Gamma_{X\times_Y X}(\mathcal{I}^{\bullet\vee} \otimes \mathcal{I}^\bullet)).$$

For open subsets $U \subseteq X \times_Y X$, the map

$$\Gamma_X(\mathcal{I}^{\bullet\vee} \otimes \mathcal{I}^\bullet)(U) \longrightarrow \Gamma_{X\times_Y X}(\mathcal{I}^{\bullet\vee} \otimes \mathcal{I}^\bullet)(U)$$

sends $\sum \alpha_i^\bullet \otimes \beta_i^\bullet$, whose support is in X, to $\sum \alpha_i^\bullet \otimes \beta_i^\bullet$ itself. The left hand side is identified with $\operatorname{Ext}^*_{K^b(X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)(U \cap X)$, where $K^b(X)$ is the homotopy category of the additive category of injective sheaves on X. On the other hand, if $U = \pi''^{-1}(V)$, for an open subset $V \subseteq Y$, then $U \cap X = g^{-1}(V)$ and

$$\Gamma_{X\times_Y X}(\mathcal{I}^{\bullet\vee} \otimes \mathcal{I}^\bullet)(U) = \Gamma_{\overline{\Delta(Y)}}(g_*\mathcal{I}^{\bullet\vee} \otimes g_*\mathcal{I}^\bullet)(V)$$

as before, so that the right hand side is identified with $\operatorname{Ext}^*_{K^b(Y)}(g_*\mathcal{I}^\bullet, g_*\mathcal{I}^\bullet)(V)$. Therefore, the pushforward ι_* of the Borel-Moore homology groups is the functorial algebra homomorphism g_*, namely $V = Y$ in the collection of maps

$$\begin{aligned} g_* : \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{K^b(g^{-1}(V))}(\mathcal{I}^\bullet|_{g^{-1}(V)}, \mathcal{I}^\bullet|_{g^{-1}(V)}[i]) \\ \longrightarrow \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{K^b(V)}(g_*\mathcal{I}^\bullet|_V, g_*\mathcal{I}^\bullet|_V[i]).\end{aligned}$$

This is $Rg_* : \operatorname{Ext}^*_{D^b(X)}(\mathcal{A}, \mathcal{A}) \to \operatorname{Ext}^*_{D^b(Y)}(Rg_*\mathcal{A}, Rg_*\mathcal{A})$ as desired. $\square$

In the following, we write $\mathrm{Ext}^*_{D^b(X)}(\mathcal{A})$ for $\mathrm{Ext}^*_{D^b(X)}(\mathcal{A},\mathcal{A})$, and we denote

$$\pi^a_{n-1,n} : p_iY^a_n \longrightarrow \mathcal{N}^a_n \times \mathbb{P}^m,$$
$$\pi^a_n : p_iY^a_n \longrightarrow \mathcal{N}^a_n.$$

We remark that $\pi^a_n = \rho_n \circ \pi^a_{n-1,n}$.

Corollary 4.9. *We have the isomorphisms*

$$p_iH^a_np_i \simeq \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(R\pi^a_{n!}\mathbb{C}),$$
$$p_iH^a_{n-1,n}p_i \simeq \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n\times\mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C})$$

such that the inclusion $p_iH^a_{n-1,n}p_i \hookrightarrow p_iH^a_np_i$ is identified with the following functorial algebra homomorphism.

$$R\rho_{n*} : \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n\times\mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C}) \longrightarrow \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(R\pi^a_{n!}\mathbb{C}).$$

Proof. Set $M = p_iY^a_n$, $X = \mathcal{N}^a_n \times \mathbb{P}^m$ and $Y = \mathcal{N}^a_n$. Then Lemma 4.8 implies the result. □

4.4. **Geometric construction of U^-_v.** Let U^-_v as in section 2. By Lusztig's theory, we may realize U^-_v geometrically by using his geometric induction and restriction functors [21]. In fact, this is essentially the Hall algebra construction which we already explained in section 2. We only need the special case which corresponds to the multiplication by f_i, which we shall explain here.

Recall that $\mathbb{C}^n$ has the eigenspace decomposition $\mathbb{C}^n = \oplus_{i\in\mathbb{Z}/e\mathbb{Z}}V_i$ with respect to $s = \mathrm{diag}(\zeta^{s_1},\ldots,\zeta^{s_n})$. We suppose that $s_n = i$.

Let $\dim V = m+1$ as before and let $W_i = V_i \cap \mathbb{C}^{n-1}$ and $W_j = V_j$, for $j \neq i$. Note that $W_i \neq V_i$. Thus we have $\mathbb{C}^{n-1} = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} W_i$. Then, we consider the diagram

$$E_W \xleftarrow{p_1} G_n(s) \times_{U_{n-1,n}(s)} F_{V,W} \xrightarrow{p_2} G_n(s) \times_{P_{n-1,n}(s)} F_{V,W} \xrightarrow{p_3} E_V,$$

where E_W, E_V and $F_{V,W}$ are defined by

$$E_W = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(W_i,W_{i+1}),\quad E_V = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(V_i,V_{i+1})$$
$$F_{V,W} = \{X \in E_V \mid XW_i \subseteq W_{i+1},\ \text{for all } i \in \mathbb{Z}/e\mathbb{Z}.\},$$

and $p_1(g,X) = X|_{\mathbb{C}^{n-1}}$, $p_2(g,X) = (g,X)$ and $p_3(g,X) = gXg^{-1}$.

We only consider those objects whose supports are contained in the null-cones. We denote the subdiagram by [2]

$$\mathcal{N}^{a;i}_{n-1} \xleftarrow{\nu_{n-1,n}} G_n(s) \times_{U_{n-1,n}(s)} \mathcal{N}^a_{n-1,n} \xrightarrow{\mu_{n-1,n}} G_n(s) \times_{P_{n-1,n}(s)} \mathcal{N}^a_{n-1,n} \xrightarrow{\rho_n} \mathcal{N}^a_n.$$

Note that $G_n(s) \times_{P_{n-1,n}(s)} \mathcal{N}^a_{n-1,n} = \{(X,U) \mid XU \subseteq U\} \subseteq \mathcal{N}^a_n \times \mathbb{P}^m$.

[2] Recall that $\mathcal{N}_{n-1,n} = \{X \in \mathcal{N}_n \mid X\mathbb{C}^{n-1} \subseteq \mathbb{C}^{n-1}\}$.

$\mathcal{N}_{n-1}^{a;i}$ has finitely many $G_{n-1}(s)$-orbits and the stabilizer group of a point in each orbit $\mathcal{O}_\varphi$, for a multisegment φ, is connected. We denote by

$$IC_\varphi = IC(\overline{\mathcal{O}_\varphi}, \mathbb{C}),$$

the intersection cohomology complex associated with the orbit $\mathcal{O}_\varphi$ and the trivial local system on it. Then, $\nu_{n-1,n}^* IC_\varphi$ is a $L_{n-1,n}(s)$-equivariant simple perverse sheaf up to degree shift, and we may write $\nu_{n-1,n}^* IC_\varphi \simeq \mu_{n-1,n}^* IC_\varphi^\flat$ up to degree shift, for some simple perverse sheaf $IC_\varphi^\flat$ on $\mathcal{N}_n^a \times \mathbb{P}^m$. $IC_\varphi^\flat$ is unique up to isomorphism. In fact, we have an integer d independent of φ, given by the difference of the dimensions of the fibers of $\mu_{n-1,n}$ and $\nu_{n-1,n}$, such that $IC_\varphi^\flat = {}^p\mathcal{H}^d(\nu_{n-1,n*}\mu_{n-1,n}^* IC_\varphi)$. We define a functor $\mathrm{Ind}_i^\flat$ by $\mathrm{Ind}_i^\flat(IC_\varphi) = IC_\varphi^\flat$. Then, we define the induction functor by

$$\mathrm{Ind}_i = R\rho_{n*} \circ \mathrm{Ind}_i^\flat .$$

Now, as in the proof of [21, 9.2.3], we consider the diagram

$$Y_{n-1}^{a;i} \longleftarrow G_n(s) \times_{U_{n-1,n}(s)} Y_{n-1,n}^a \longrightarrow G_n(s) \times_{P_{n-1,n}(s)} Y_{n-1,n}^a = p_i Y_n^a,$$

which "covers" the left three terms of the above diagram with cartesian squares. We denote the leftmost vertical map by

$$\pi_{n-1}^{a;i} : Y_{n-1}^{a;i} \longrightarrow \mathcal{N}_{n-1}^{a;i}.$$

Then, we have the following equalities up to degree shift.

$$\mathrm{Ind}_i^\flat(R\pi_{n-1!}^{a;i}\mathbb{C}) = R\pi_{n-1,n!}^a\mathbb{C}, \quad \mathrm{Ind}_i(R\pi_{n-1!}^{a;i}\mathbb{C}) = R\pi_{n!}^a\mathbb{C}.$$

The main result of [21] is the geometric construction of the algebra U_v^- in terms of the induction functor. The simple perverse sheaves IC_φ are part of the canonical basis and Ind_i corresponds the multiplication from the left by f_i. The canonical basis defines the crystal $B(\infty)$. Combined with Kashiwara's result [14, Proposition 6.2.3], we have the following. [3]

Lemma 4.10.
(1) *Let φ be a multisegment of size $n-1$. Then, we may write*

$$\mathrm{Ind}_i(IC_\varphi) = \sum_{j=0}^{\epsilon_i(\varphi)} IC_{\tilde{f}_i\varphi}[\epsilon_i(\varphi) - 2j] + \sum_{j\in\mathbb{Z}} R_{\varphi,j}[j],$$

for certain perverse sheaves $R_{\varphi,j}$ on $\mathcal{N}_n^a$. [4]
(2) *Suppose that IC_ψ, for a multisegment ψ of size n, appears in $R_{\varphi,j}$, for some j. Then, we have*

$$-\epsilon_i(\psi) + 2 \le j \le \epsilon_i(\psi) - 2.$$

[3] It is known that [14, Proposition 6.2.3] may be proved in this geometric framework.
[4] The summation means the direct sum.

4.5. **Some semisimple quotients.** Lemma 3.10 implies that the surjection $p_i H^a_{n-1,n} p_i \to H^{a;i}_{n-1}$ induces the identity map

$$\begin{aligned} p_i H^a_{n-1,n} p_i / \operatorname{Rad}(p_i H^a_{n-1,n} p_i) &\simeq \oplus_M \operatorname{End}_{\mathbb{C}}(M) \\ &\longrightarrow \oplus_M \operatorname{End}_{\mathbb{C}}(M) \simeq H^{a;i}_{n-1} / \operatorname{Rad}(H^{a;i}_{n-1}), \end{aligned}$$

where M runs through the common complete set of isomorphism classes of simple modules.

On the other hand, the complete set of isomorphism classes of simple modules of $\operatorname{Ext}^*_{D^b(\mathcal{N}^a_n \times \mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C})$ and $\operatorname{Ext}^*_{D^b(\mathcal{N}^{a;i}_{n-1})}(R\pi^{a;i}_{n-1!}\mathbb{C})$ may be described by those simple perverse sheaves that appear in $R\pi^a_{n-1,n!}\mathbb{C}$ and $R\pi^{a;i}_{n-1!}\mathbb{C}$ after some shift, respectively. The degree of the shift depends on the perverse sheaf. As they are semisimple complexes by the decomposition theorem, we write

$$R\pi^a_{n-1,n!}\mathbb{C} \simeq \sum_{\psi} \sum_{m \in \mathbb{Z}} IC_\psi[m]^{\oplus m_{\psi,m}}, \quad R\pi^{a;i}_{n-1!}\mathbb{C} \simeq \sum_{\varphi} \sum_{m \in \mathbb{Z}} IC_\varphi[m]^{\oplus n_{\varphi,m}},$$

where IC_ψ and IC_φ are simple perverse sheaves on $\mathcal{N}^a_n \times \mathbb{P}^m$ and $\mathcal{N}^{a;i}_{n-1}$, respectively. Let $L_{\psi,m} = \mathbb{C}^{m_{\psi,m}}$ and $L_{\varphi,m} = \mathbb{C}^{m_{\varphi,m}}$ be the multiplicity spaces of $IC_\psi[m]$ and $IC_\varphi[m]$, respectively. Define

$$L_\psi = \bigoplus_{m \in \mathbb{Z}} L_{\psi,m}, \quad L_\varphi = \bigoplus_{m \in \mathbb{Z}} L_{\varphi,m}.$$

Then, we have

$$\begin{aligned} \operatorname{Ext}^*_{D^b(\mathcal{N}^a_n \times \mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C}) &\simeq \bigoplus_{\psi',\psi''} \operatorname{Ext}^*(IC_{\psi'}, IC_{\psi''}) \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(L_{\psi'}, L_{\psi''}), \\ \operatorname{Ext}^*_{D^b(\mathcal{N}^{a;i}_{n-1})}(R\pi^{a;i}_{n-1!}\mathbb{C}) &\simeq \bigoplus_{\varphi',\varphi''} \operatorname{Ext}^*(IC_{\varphi'}, IC_{\varphi''}) \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(L_{\varphi'}, L_{\varphi''}). \end{aligned}$$

In other words, $\operatorname{Ext}^*_{D^b(\mathcal{N}^a_n \times \mathbb{P}^m)}(R\pi^a_{n-1,n!}\mathbb{C})$ is the matrix algebra which has block partitions of rows and columns such that the blocks are labelled by ψ and the entries in the (ψ'', ψ') component are elements of $\operatorname{Ext}^*(IC_{\psi'}, IC_{\psi''})$. In particular, its semisimple quotient is the block diagonal matrix algebra such that the entries of the (ψ, ψ)-component are

$$\operatorname{Ext}^{\geq 0}(IC_\psi, IC_\psi) / \operatorname{Ext}^{>0}(IC_\psi, IC_\psi) \simeq \mathbb{C}.$$

We have the similar matrix algebra description for $\operatorname{Ext}^*_{D^b(\mathcal{N}^{a;i}_{n-1})}(R\pi^{a;i}_{n-1!}\mathbb{C})$ as well.

4.6. **A key result.** We prove Theorem 4.12, which we will need in the geometric proof of the modular branching rule in the next subsection.

Define $\eta_{n-1,n} : \mathcal{N}^a_{n-1,n} \hookrightarrow \mathcal{N}^a_n \times \mathbb{P}^m$, $\kappa_{n-1,n} : \mathcal{N}^a_{n-1,n} \to \mathcal{N}^{a;i}_{n-1}$. We identify $\mathcal{N}^{a;i}_{n-1}$ with the zero section of $\kappa_{n-1,n}$ and we obtain the closed embedding

$$\epsilon_{n-1,n} : \mathcal{N}^{a;i}_{n-1} \hookrightarrow \mathcal{N}^a_n \times \mathbb{P}^m.$$

$\eta_{n-1,n}^* R\pi_{n-1,n!}^a\mathbb{C}$ is the pushforward of the constant sheaf on $Y_{n-1,n}^a$ to $\mathcal{N}_{n-1,n}^a$, and we have the following cartesian diagram.

$$\begin{array}{ccc} Y_{n-1,n}^a & \to & Y_{n-1}^{a;i} \\ \downarrow & & \downarrow \\ \mathcal{N}_{n-1,n}^a & \to & \mathcal{N}_{n-1}^{a;i} \end{array}$$

Thus, $\eta_{n-1,n}^* R\pi_{n-1,n!}^a\mathbb{C} \simeq \kappa_{n-1,n}^* R\pi_{n-1!}^{a;i}\mathbb{C}$ and we conclude that

$$\epsilon_{n-1,n}^* R\pi_{n-1,n!}^a\mathbb{C} \simeq R\pi_{n-1!}^{a;i}\mathbb{C}.$$

Lemma 4.11. *We consider the functorial algebra homomorphism*

$$\epsilon_{n-1,n}^* : \operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1,n!}^a\mathbb{C}) \longrightarrow \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1!}^{a;i}\mathbb{C}).$$

Then, it induces the isomorphism

$$\begin{aligned} &\operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1,n!}^a\mathbb{C})/\operatorname{Rad}(\operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1,n!}^a\mathbb{C})) \\ &\quad\simeq \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1!}^{a;i}\mathbb{C})/\operatorname{Rad}(\operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1!}^{a;i}\mathbb{C})) \end{aligned}$$

and it is identified with the identity map

$$p_iH_{n-1,n}^a p_i/\operatorname{Rad}(p_iH_{n-1,n}^a p_i) \simeq H_{n-1}^{a;i}/\operatorname{Rad}(H_{n-1}^{a;i}).$$

Further, its inverse is induced by the functorial algebra homomorphism

$$\operatorname{Ind}_i^\flat : \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1!}^{a;i}\mathbb{C}) \longrightarrow \operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1,n!}^a\mathbb{C}).$$

Proof. Note that

$$\operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1!}^{a;i}\mathbb{C}) = \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(\oplus_\varphi IC_\varphi \otimes_\mathbb{C} L_\varphi)$$

as $\mathbb{C}$-algebras. Thus, the functorial algebra homomorphism

$$\operatorname{Ind}_i^\flat : \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(\oplus_\varphi IC_\varphi \otimes_\mathbb{C} L_\varphi) \to \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1,n}^{a})}(\oplus_\varphi IC_\varphi^\flat \otimes_\mathbb{C} L_\varphi)$$

induces the identity map

$$\begin{aligned} &\operatorname{Ext}^0_{D^b(\mathcal{N}_{n-1}^{a;i})}(\oplus_\varphi IC_\varphi \otimes_\mathbb{C} L_\varphi) = \oplus_\varphi \operatorname{End}_\mathbb{C}(L_\varphi) \\ &\qquad\qquad \longrightarrow \oplus_\varphi \operatorname{End}_\mathbb{C}(L_\varphi) = \operatorname{Ext}^0_{D^b(\mathcal{N}_{n-1,n}^{a})}(\oplus_\varphi IC_\varphi^\flat \otimes_\mathbb{C} L_\varphi). \end{aligned}$$

That is, $\operatorname{Ind}_i^\flat$ induces the isomorphism

$$\begin{aligned} &\operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1!}^{a;i}\mathbb{C})/\operatorname{Rad}(\operatorname{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(R\pi_{n-1!}^{a;i}\mathbb{C})) \\ &\quad\simeq \operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1,n!}^{a}\mathbb{C})/\operatorname{Rad}(\operatorname{Ext}^*_{D^b(\mathcal{N}_{n-1}^{a;i})}(R\pi_{n-1,n!}^{a}\mathbb{C})). \end{aligned}$$

it is identified with the identity map

$$H_{n-1}^{a;i}/\operatorname{Rad}(H_{n-1}^{a;i}) \simeq p_iH_{n-1,n}^a p_i/\operatorname{Rad}(p_iH_{n-1,n}^a p_i).$$

On the other hand, we have $\mathrm{Ind}_i^\flat(R\pi^{a;i}_{n-1!}\mathbb{C}) \simeq R\pi^a_{n-1,n!}\mathbb{C}$ up to degree shift, and $\epsilon^*_{n-1,n}R\pi^a_{n-1,n!}\mathbb{C} \simeq R\pi^{a;i}_{n-1!}\mathbb{C}$. Thus, $\mathrm{Ind}_i^\flat$ and $\epsilon^*_{n-1,n}$ are inverse to the other on the semisimple quotients, and the claim follows. □

Theorem 4.12. *Consider the functorial algebra homomorphism*

$$\mathrm{Ind}_i : \mathrm{Ext}^*_{D^b(\mathcal{N}^{a;i}_{n-1})}(R\pi^{a;i}_{n-1!}\mathbb{C}) \longrightarrow \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(R\pi^a_{n!}\mathbb{C}).$$

If M is a simple H^a_n-module, then the action of $H^{a;i}_{n-1}$ on $\mathrm{Top}(p_iM)$ coincides with that given by Ind_i under the identification

$$H^{a;i}_{n-1} = \mathrm{Ext}^*_{D^b(\mathcal{N}^{a;i}_{n-1})}(R\pi^{a;i}_{n-1!}\mathbb{C}),\quad p_iH^a_np_i = \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(R\pi^a_{n!}\mathbb{C}).$$

Proof. Let $(Y^a_n)_\nu$ be the set of (X, F) such that the flag type is ν, as before. We denote $\pi_{n,\nu} : (Y^a_n)_\nu \to \mathcal{N}^a_n$ and

$$\mathcal{M}_\nu = \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(R\pi_{n,\nu!}\mathbb{C}).$$

Then, by our identification, we have

$$H^a_n = \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(\oplus_\nu\mathcal{M}_\nu)$$

where ν runs through flag types which are permutations of $(s_1, \ldots, s_n)$. Write

$$\bigoplus_\nu \mathcal{M}_\nu = \bigoplus_\psi IC_\psi \otimes_{\mathbb{C}} L_\psi.$$

Then, $H^a_n = \bigoplus_{\psi',\psi''}\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_{\psi'}, IC_{\psi''})\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(L_{\psi'}, L_{\psi''})$ and we view it as the block partitioned matrix algebra whose entries of the (ψ'', ψ')-component are elements of $\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_{\psi'}, IC_{\psi''})$. Define

$$P_\psi = \bigoplus_{\psi'}\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC_{\psi'}) \otimes_{\mathbb{C}} L_{\psi'}.$$

Then, it is a direct summand of H^a_n and we view it as the space of block partitioned column vectors whose entries in the block $L_{\psi'}$ are elements of $\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC_{\psi'})$.

$$\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_{\psi'}, IC_{\psi''})\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(L_{\psi'}, L_{\psi''})\times\mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC_{\psi'})\otimes_{\mathbb{C}}L_{\psi'} \longrightarrow \mathrm{Ext}^*_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC_{\psi''}) \otimes_{\mathbb{C}} L_{\psi''}$$

shows that P_ψ is a left ideal of H^a_n so that it is a projective H^a_n-module. It is clear that

$$L_\psi = \frac{\mathrm{Ext}^{\geq 0}_{D^b(\mathcal{N}^a_n)}(IC_\psi, \oplus_\nu\mathcal{M}_\nu)}{\mathrm{Ext}^{>0}_{D^b(\mathcal{N}^a_n)}(IC_\psi, \oplus_\nu\mathcal{M}_\nu)}$$

is a simple H^a_n-module or zero and that any simple H^a_n-module appears in this way. Thus, we assume that $M = L_\psi$. Then, Lemma 4.6 says that multiplication by p_i amounts to picking out the connected components $p_iY_n^a$ so that

$$p_iL_\psi = \frac{\mathrm{Ext}^{\geq 0}_{D^b(\mathcal{N}_n^a)}(IC_\psi, \oplus_\nu \mathcal{M}_\nu)}{\mathrm{Ext}^{>0}_{D^b(\mathcal{N}_n^a)}(IC_\psi, \oplus_\nu \mathcal{M}_\nu)}$$

where ν runs through permutations of $(s_1, \ldots, s_n)$ such that $\nu_n = i$. Suppose that $p_iL_\psi \neq 0$. It is a simple $p_iH_n^ap_i$-module. Let $\pi_{n-1,n,\nu} : (Y_n^a)_\nu \to \mathcal{N}_n^a \times \mathbb{P}^m$ and

$$\mathcal{M}_\nu^\flat = \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(R\pi_{n-1,n,\nu!}\mathbb{C}).$$

Then $p_iH_{n-1,n}^ap_i = \mathrm{Ext}_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(\oplus_\nu\mathcal{M}_\nu^\flat)$, where ν runs through permutations of $(s_1, \ldots, s_n)$ such that $\nu_n = i$, and it acts on p_iL_ψ through $R\rho_{n*}$ by Corollary 4.9. Now, we consider $\mathrm{Top}(p_iL_\psi)$. Then, the action of $p_iH_{n-1,n}^ap_i$ factors through $H_{n-1}^{a;i}/\operatorname{Rad}(H_{n-1}^{a;i})$ and Lemma 4.11 implies that it is given by $\mathrm{Ind}_i^\flat$. We have proved that the action of $H_{n-1}^{a;i}$ on $\mathrm{Top}(p_iL_\psi)$ coincides with the action of $H_{n-1}^{a;i}$ given by the functorial algebra homomorphism Ind_i. □

4.7. **The geometric proof.** Having proved Theorem 4.12, we are now able to give the promised geometric proof of the modular branching rule. We write each simple H_n^a-module as in the proof of the above theorem

$$L_\psi = \frac{\mathrm{Ext}^{\geq 0}_{D^b(\mathcal{N}_n^a)}(IC_\psi, \oplus_\nu \mathcal{M}_\nu)}{\mathrm{Ext}^{>0}_{D^b(\mathcal{N}_n^a)}(IC_\psi, \oplus_\nu \mathcal{M}_\nu)}.$$

Suppose that $p_iL_\psi \neq 0$. We want to show that $\mathrm{Top}(p_iL_\psi)$ contains $L_{\tilde{e}_i\psi}$. As the simple $H_{n-1}^{a;i}$-modules are the same as the simple $p_iH_{n-1,n}^ap_i$-modules, we consider the restriction of p_iL_ψ to $p_iH_{n-1,n}^ap_i$. Let $\pi_{n,\nu}^a = \rho_n \circ \pi_{n-1,n,\nu}^a$. Then, we have

$$R\pi_{n!}^a\mathbb{C} = \bigoplus_{\nu \text{ such that } \nu_n = i} R\pi_{n,\nu!}^a\mathbb{C},$$

which is equal to $\mathrm{Ind}_i(R\pi_{n-1!}^{a;i}\mathbb{C})$ up to degree shift. Thus, we write

$$\bigoplus_{\nu \text{ such that } \nu_n = i} \mathcal{M}_\nu^\flat = \bigoplus_\varphi IC_\varphi^\flat \otimes_\mathbb{C} L_\varphi$$

and restrict the action of $p_iH_n^ap_i$ on p_iL_ψ to $p_iH_{n-1,n}^ap_i$ through $R\rho_{n*}$, which is the functorial algebra homomorphism given by

$$p_iH_{n-1,n}^ap_i = \bigoplus_{\varphi',\varphi''} \mathrm{Ext}^*_{D^b(\mathcal{N}_n^a\times\mathbb{P}^m)}(\mathrm{Ind}_i^\flat IC_{\varphi'}, \mathrm{Ind}_i^\flat IC_{\varphi''}) \bigotimes_\mathbb{C} \mathrm{Hom}_\mathbb{C}(L_{\varphi'}, L_{\varphi''})$$

$$\longrightarrow \bigoplus_{\varphi',\varphi''} \mathrm{Ext}^*_{D^b(\mathcal{N}_n^a)}(\mathrm{Ind}_i IC_{\varphi'}, \mathrm{Ind}_i IC_{\varphi''}) \bigotimes_\mathbb{C} \mathrm{Hom}_\mathbb{C}(L_{\varphi'}, L_{\varphi''}) = p_iH_n^ap_i.$$

To study this, we introduce a block algebra description of $p_iH^a_{n-1,n}p_i$-action on p_iL_ψ. As

$$\bigoplus_{\nu \text{ such that } \nu_n=i} \mathcal{M}_\nu = \bigoplus_\varphi \left(IC^{\oplus(\epsilon_i(\varphi)+1)}_{\tilde f_i\varphi} + \sum_j R_{\varphi,j} \right) \otimes_{\mathbb{C}} L_\varphi,$$

by Lemma 4.10(1), p_iL_ψ has the decomposition

$$p_iL_\psi = \bigoplus_\varphi \frac{\mathrm{Ext}^{\geq 0}_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC^{\oplus(\epsilon_i(\varphi)+1)}_{\tilde f_i\varphi} + \sum_j R_{\varphi,j})}{\mathrm{Ext}^{>0}_{D^b(\mathcal{N}^a_n)}(IC_\psi, IC^{\oplus(\epsilon_i(\varphi)+1)}_{\tilde f_i\varphi} + \sum_j R_{\varphi,j})} \otimes_{\mathbb{C}} L_\varphi.$$

Thus, we have the corresponding block decomposition of $\mathrm{End}_{\mathbb{C}}(p_iL_\psi)$.

Observe that IC_ψ appears in $R_{\varphi,j}$ only if $\epsilon_i(\varphi) < \epsilon_i(\tilde e_i\psi)$ and IC_ψ appears in $IC^{\oplus(\epsilon_i(\varphi)+1)}_{\tilde f_i\varphi}$ only if $\varphi = \tilde e_i\psi$. Hence, only those blocks L_φ with $\epsilon_i(\varphi) < \epsilon_i(\tilde e_i\psi)$ and $L_{\tilde e_i\psi}$ appear in the above block decomposition.

To obtain the (φ'', φ')-component of the representation of $p_iH^a_{n-1,n}p_i$ on p_iL_ψ, we consider the image of $\mathrm{Ext}^k_{D^b(\mathcal{N}^a_n\times\mathbb{P}^m)}(IC^\flat_{\varphi'}, IC^\flat_{\varphi''})$, for $k \geq 0$, through the action of

$$\mathrm{Ext}^k_{D^b(\mathcal{N}^a_n)}(\mathrm{Ind}_i(IC_{\varphi'}), \mathrm{Ind}_i(IC_{\varphi''})).$$

The image may be nonzero only when $IC_\psi[j']$, for some $j' \in \mathbb{Z}$, appears in $\mathrm{Ind}_i(IC_{\varphi'})$ and $IC_\psi[j'']$, for some $j'' \in \mathbb{Z}$, appears in $\mathrm{Ind}_i(IC_{\varphi''})$ such that $-j' + j'' + k = 0$. In particular, $j'' \leq j'$ is necessary. Since

$$\begin{aligned}
\mathrm{Ind}_i(IC_{\varphi'}) &= \sum_{j'=0}^{\epsilon_i(\varphi')} IC_{\tilde f_i\varphi'}[\epsilon_i(\varphi') - 2j'] + \sum_{j'\in\mathbb{Z}} R_{\varphi',j'}[j'],\\
\mathrm{Ind}_i(IC_{\varphi''}) &= \sum_{j''=0}^{\epsilon_i(\varphi'')} IC_{\tilde f_i\varphi''}[\epsilon_i(\varphi'') - 2j''] + \sum_{j''\in\mathbb{Z}} R_{\varphi'',j''}[j''],
\end{aligned}$$

there are four cases to consider.

- Suppose that $\varphi' = \varphi'' = \tilde e_i\psi$. We number the rows and columns of the block matrix by $0 \leq j'', j' \leq \epsilon_i(\psi) - 1$ such that $\epsilon_i(\psi) - 1 - 2j''$ and $\epsilon_i(\psi) - 1 - 2j'$ are increasing. Then, the entries may be nonzero only when $\epsilon_i(\psi) - 1 - 2j'' \leq \epsilon_i(\psi) - 1 - 2j'$. Thus, we obtain an upper block triangular matrix whose diagonal block components are $\mathrm{End}_{\mathbb{C}}(L_{\tilde e_i\psi})$.
- Suppose that $\varphi' \neq \tilde e_i\psi = \varphi''$. We number the rows as before, and the columns such that j' is increasing. If IC_ψ appears in $L_{\varphi',j'}$ then the entries may be nonzero only when $\epsilon_i(\psi) - 1 - 2j'' \leq j'$. Hence, each row has entries only after the column number $\epsilon_i(\psi) - 1 - 2j''$. Now, Lemma 4.10(2) implies that $j' \leq \epsilon_i(\psi) - 2$ so that $j'' = 0$ cannot happen. Hence, all the entries of the last row are zero.

- Suppose that $\varphi' = \tilde{e}_i\psi \neq \varphi''$. Then, each column has entries only before some column number.
- Suppose that $\varphi' \neq \tilde{e}_i\psi$ and $\varphi'' \neq \tilde{e}_i\psi$. Then we have an upper block triangular matrix again.

The first two cases show that there is a $p_iH^a_{n-1,n}p_i$-submodule L'_ψ of L_ψ such that $L_\psi/L'_\psi \simeq L_{\tilde{e}_i\psi}$. Thus, $L_{\tilde{e}_i\psi}$ appears in $\mathrm{Top}(p_iL_\psi)$. Now, following [17], Grojnowski and Vazirani proved in Vazirani's thesis that $\mathrm{Soc}(p_iL_\psi)$ is simple [9]. By Specht module theory, the simple modules are self-dual so that $\mathrm{Top}(p_iL_\psi)$ is isomorphic to $\mathrm{Soc}(p_iL_\psi)$. Thus, we have proved that $\mathrm{Soc}(p_iL_\psi) = L_{\tilde{e}_i\psi}$. Thus, Theorem 4.2 and Theorem 4.4 follow.

5. Crystals of deformed Fock spaces

In this section, we recall results on deformed Fock spaces which are related to the combinatorial construction of simple H_n-modules.

5.1. **Crystals of deformed Fock spaces.** Let $l \in \mathbb{Z}_{>0}$ and we choose a *multicharge*

$$\mathbf{v} = (\mathrm{v}_0, ..., \mathrm{v}_{l-1}) \in \mathbb{Z}^l.$$

We denote $\mathrm{v}_i + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$ by $\overline{\mathrm{v}_i}$, for $1 \leq i \leq l$. Let Λ_i, for $i \in \mathbb{Z}/e\mathbb{Z}$, be the fundamental weights of $\mathfrak{g}$, and define a dominant weight Λ by

$$\Lambda = \Lambda_{\overline{\mathrm{v}_0}} + \cdots + \Lambda_{\overline{\mathrm{v}_{l-1}}}.$$

We consider various multicharges which give a fixed Λ.

Let $V_v(\Lambda)$ be the integrable highest weight $U_v(\mathfrak{g})$-module of highest weight Λ. We want to realize $V_v(\Lambda)$ as a $U_v(\mathfrak{g})$-submodule of the level l deformed Fock space $\mathcal{F}^{\mathbf{v}}$ associated with the multicharge $\mathbf{v}$.

As a $\mathbb{C}(v)$-vector space, the level l Fock space $\mathcal{F}^{\mathbf{v}}$ admits the set of all l-partitions as a natural basis. Namely, the underlying vector space is

$$\mathcal{F} = \bigoplus_{n\geq 0} \bigoplus_{\boldsymbol{\lambda}\in\Pi_{l,n}} \mathbb{C}(v)\boldsymbol{\lambda},$$

where $\Pi_{l,n}$ is the set of l-partitions of rank n. We do not give explicit formulas to define the $U_v(\mathfrak{g})$-module structure on $\mathcal{F}^{\mathbf{v}}$, but it is defined in terms of the total order $\prec_{\mathbf{v}}$ introduced below. The action we adopt here is the one which was introduced by Jimbo, Misra, Miwa and Okado in [11]. Let

$$L^{\mathbf{v}} = \bigoplus_{n\geq 0} \bigoplus_{\boldsymbol{\lambda}\in\Pi_{l,n}} R\boldsymbol{\lambda}, \quad B^{\mathbf{v}} = \bigsqcup_{n\geq 0} \Pi_{l,n}.$$

Then, $(L^{\mathbf{v}}, B^{\mathbf{v}})$ is a crystal basis of $\mathcal{F}^{\mathbf{v}}$. In this article, it suffices to recall the crystal structure on the set of l-partitions. Before doing this, we explain basic terminology on l-partitions.

Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$ be an l-partition, which is identified with the corresponding l-tuple of Young diagrams. Then, we can speak of nodes of $\boldsymbol{\lambda}$, which are nodes of the Young diagrams. We identify a node γ of $\boldsymbol{\lambda}$ with a triplet (a,b,c) where $c \in \{0,...,l-1\}$ is such that γ is a node of $\lambda^{(c)}$, and a and b are the row and the column indices of the node γ in $\lambda^{(c)}$, respectively.

Definition 5.1. Let $\gamma = (a,b,c)$ be a node of an l-partition $\boldsymbol{\lambda}$. Then, its *content* $c(\gamma)$ and *residue* $\mathrm{r}es(\gamma)$ are defined by

$$c(\gamma) = b - a + \mathrm{v}_c \in \mathbb{Z} \ \text{ and } \ \mathrm{r}es(\gamma) = \overline{c(\gamma)} \in \mathbb{Z}/e\mathbb{Z},$$

respectively.

Let γ be a node of $\boldsymbol{\lambda}$. Then we say that γ is an *i-node*, for $i \in \mathbb{Z}/e\mathbb{Z}$, if $\mathrm{r}es(\gamma) = i$. Suppose that $\boldsymbol{\lambda}\backslash\{\gamma\}$ is again an l-partition, which we denote by $\boldsymbol{\mu}$. Then, we say that γ is a *removable i-node* of $\boldsymbol{\lambda}$ and γ is an *addable i-node* of $\boldsymbol{\mu}$. We introduce a total order $\prec_{\mathbf{v}}$ on the set of addable and removable i-nodes of an l-partition $\boldsymbol{\lambda}$.

Definition 5.2. Let $\gamma_1 = (a_1,b_1,c_1)$ and $\gamma_2 = (a_2,b_2,c_2)$ be i-nodes of $\boldsymbol{\lambda}$. We define the order $\prec_{\mathbf{v}}$ by

$$\gamma_1 \prec_{\mathbf{v}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2), \ \text{or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2. \end{cases}$$

The order $\prec_{\mathbf{v}}$ depends on the choice of the multicharge $\mathbf{v}$ when $l > 1$.

Now, we can explain the crystal structure on $B^{\mathbf{v}}$, which is defined by the total order $\prec_{\mathbf{v}}$. Let $\boldsymbol{\lambda}$ be an l-partition as above. Let $N_i(\boldsymbol{\lambda})$, for $i \in \mathbb{Z}/e\mathbb{Z}$, be the number of i-nodes of $\boldsymbol{\lambda}$. Then we define

$$\mathrm{wt}(\boldsymbol{\lambda}) = \Lambda - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} N_i(\boldsymbol{\lambda})\alpha_i.$$

The rule to compute $\tilde{e}_i\boldsymbol{\lambda}$ is as follows. The rule to compute $\tilde{f}_i\boldsymbol{\lambda}$ is similar.

> We read addable and removable i-nodes of $\boldsymbol{\lambda}$ in the increasing order with respect to $\prec_{\mathbf{v}}$. Then we delete a consecutive pair of a removable i-node and an addable i-node in this order as many as possible. We call this procedure RA deletion.
> - If there remains no removable i-node, define $\tilde{e}_i\boldsymbol{\lambda} = 0$.
> - Otherwise, we call the leftmost removable i-node, say γ, the *good i-node of* $\boldsymbol{\lambda}$, and define $\tilde{e}_i\boldsymbol{\lambda} = \boldsymbol{\lambda}\backslash\{\gamma\}$.

Finally, we define

$$\epsilon_i(\boldsymbol{\lambda}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k\boldsymbol{\lambda} \neq 0\}, \quad \varphi_i(\boldsymbol{\lambda}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k\boldsymbol{\lambda} \neq 0\}.$$

The empty l-partition $\boldsymbol{\emptyset} = (\emptyset,...,\emptyset)$ is a highest weight vector of weight Λ in $\mathcal{F}^{\mathbf{v}}$. We denote by $V_v(\mathbf{v})$ the $U_v(\mathfrak{g})$-submodule generated by $\boldsymbol{\emptyset}$. Then, $V_v(\mathbf{v})$ is isomorphic to $V_v(\Lambda)$ as $U_v(\mathfrak{g})$-modules.

Definition 5.3. The crystal $B(\mathbf{v})$ is the connected subcrystal of $B^{\mathbf{v}}$ that contains the empty l-partition $\boldsymbol{\emptyset}$. An l-partition in $B(\mathbf{v})$ is called an *Uglov l-partition* of multicharge $\mathbf{v}$.

As $B(\mathbf{v})$ is the subcrystal which corresponds to $V_v(\mathbf{v})$, it is isomorphic to the highest weight crystal $B(\Lambda)$.

5.2. **FLOTW l-partitions.** Define a set $\mathcal{V}_l$ of multicharges by

$$\mathcal{V}_l = \{\mathbf{v} = (\mathrm{v}_0, \dots, \mathrm{v}_{l-1}) \mid \mathrm{v}_0 \le \cdots \le \mathrm{v}_{l-1} < \mathrm{v}_0 + e\}.$$

For each l-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(l-1)})$, let $\lambda^{(c)}_j$, for $j = 1, 2, \cdots$, be the parts of $\lambda^{(c)}$. If $\lambda^{(c)}_j > 0$ then we denote the residue of the right end node of the j^{th} row of $\lambda^{(c)}$ by $\mathrm{r}es(\lambda^{(c)}_j)$, which is the residue of $\lambda^{(c)}_j - j + \mathrm{v}_c$.

Definition 5.4. Suppose that $\mathbf{v} \in \mathcal{V}_l$. A *FLOTW l-partition* of multicharge $\mathbf{v}$ is an l-partition $\boldsymbol{\lambda}$ which satifies the following two conditions.

(i) We have the inequalities

$$\lambda^{(c)}_j \ge \lambda^{(c+1)}_{j+\mathrm{v}_{c+1}-\mathrm{v}_c}, \text{ for } 0 \le c \le l-2, \text{ and } \lambda^{(l-1)}_j \ge \lambda^{(0)}_{j+e+\mathrm{v}_0-\mathrm{v}_{l-1}}.$$

(ii) For each $k \in \mathbb{Z}_{>0}$, we have

$$\{\mathrm{r}es(\lambda^{(c)}_j) \mid \lambda^{(c)}_j = k\} \ne \mathbb{Z}/e\mathbb{Z}.$$

We denote by $\Phi(\mathbf{v})_n$ the set of FLOTW l-partitions of multicharge $\mathbf{v}$ and rank n. Then, we define

$$\Phi(\mathbf{v}) = \bigsqcup_{n\ge 0} \Phi(\mathbf{v})_n, \quad \text{and} \quad \Phi = \bigsqcup_{\mathbf{v}\in\mathcal{V}_l} \Phi(\mathbf{v}).$$

We have the following result [7].

Proposition 5.5. *Suppose that* $\mathbf{v} \in \mathcal{V}_l$. *Then,* $B(\mathbf{v}) = \Phi(\mathbf{v})$.

5.3. **Kleshchev l-partitions.** If $l = 1$ then we have the level 1 deformed Fock spaces $\mathcal{F}^{\mathrm{v}}$, for $\mathrm{v} \in \mathbb{Z}$. We consider the tensor product

$$\mathcal{F}^{\mathrm{v}_{l-1}} \otimes \cdots \otimes \mathcal{F}^{\mathrm{v}_0},$$

for a multicharge $\mathbf{v}$. Note that it depends only on $\overline{\mathbf{v}} = (\overline{\mathrm{v}}_0, \dots, \overline{\mathrm{v}}_{l-1})$. Then,

$$(L^{\mathrm{v}_{l-1}} \otimes \cdots \otimes L^{\mathrm{v}_0}, B^{\mathrm{v}_{l-1}} \otimes \cdots \otimes B^{\mathrm{v}_0})$$

is a crystal basis of $\mathcal{F}^{\mathrm{v}_{l-1}} \otimes \cdots \otimes \mathcal{F}^{\mathrm{v}_0}$.

Definition 5.6. A *Kleshchev l-partition* is an l-partition $\boldsymbol{\lambda}$ such that the tensor product of the transpose of $\lambda^{(i)}$'s in the reversed order

$${}^t\lambda^{(l-1)} \otimes \cdots \otimes {}^t\lambda^{(0)}$$

belongs the connected component of $B^{\mathrm{v}_{l-1}} \otimes \cdots \otimes B^{\mathrm{v}_0}$ that contains $\emptyset \otimes \cdots \otimes \emptyset$.

We denote by Φ^K_n the set of Kleshchev l-partitions of rank n. Then we define

$$\Phi^K = \bigsqcup_{n\ge 0} \Phi^K_n.$$

We need the transpose of partitions in the definition in order to make it compatible with Specht module theory of cyclotomic Hecke algebras, which we introduce later. Note that if $\boldsymbol{\lambda}$ is Kleshchev then each component $\lambda^{(j)}$ is e-restricted.

Φ^K inherits the crystal structure from $B^{\mathrm{v}_{l-1}} \otimes \cdots \otimes B^{\mathrm{v}_0}$, and Φ^K is isomorphic to the highest weight crystal $B(\Lambda)$, again.

5.4. **Crystal isomorphisms.** As $\Phi(\mathbf{v})$ and Φ^K are isomorphic, we have a unique isomorphism of crystals between them, which we denote by

$$\Gamma : \Phi(\mathbf{v}) \to \Phi^K.$$

We may compute this bijection explicitly. In fact, if we fix n and choose another multicharge $\mathbf{w}$ such that

- w_j is sufficiently smaller than w_{j+1}, for $0 \le j \le l-2$, and
- $\overline{\mathrm{v}}_j = \overline{\mathrm{w}}_j$, for $0 \le j \le l-1$,

then the bijection between $\Phi^K_{\le n}$ and $B(\mathbf{w})_{\le n}$ given by

$$(\lambda^{(0)}, \dots, \lambda^{(l-1)}) \mapsto ({}^t\lambda^{(0)}, \dots, {}^t\lambda^{(l-1)})$$

is compatible with the crystal structures on $\Phi^K_{\le n}$ and $B(\mathbf{w})_{\le n}$. Hence, it suffices to compute the crystal isomorphism between $B(\mathbf{v})$ and $B(\mathbf{w})$.

Let $\widehat{\mathfrak{S}}_n = e\mathbb{Z} \wr \mathfrak{S}_n \subseteq \mathrm{Aut}(\mathbb{Z}^l)$ be the extended affine symmetric group. Define $\sigma_j \in \mathrm{Aut}(\mathbb{Z}^l)$, for $0 \le j \le l-2$, by

$$\sigma_j(\mathrm{v}_0, \dots, \mathrm{v}_{j-1}, \mathrm{v}_j, \dots, \mathrm{v}_{l-1}) = (\mathrm{v}_0, \dots, \mathrm{v}_j, \mathrm{v}_{j-1}, \dots, \mathrm{v}_{l-1})$$

and define $\tau \in \mathrm{Aut}(\mathbb{Z}^l)$ by $\tau(\mathrm{v}_0, \dots, \mathrm{v}_{l-1}) = (\mathrm{v}_1, \dots, \mathrm{v}_{l-1}, \mathrm{v}_0 + e)$. Then, $\widehat{\mathfrak{S}}_n$ is generated by these elements.

The following theorem was proved by the second and the third authors in [13]. As the multicharges $\mathbf{v}$ and $\mathbf{w}$ are in the same $\widehat{\mathfrak{S}}_n$-orbit, it allows us to compute the crystal isomorphism between $B(\mathbf{v})$ and $B(\mathbf{w})$ explicitly.

Theorem 5.7.
(1) *The crystal isomorphism $B(\mathbf{v}) \to B(\tau\mathbf{v})$ is given by*

$$(\lambda^{(0)}, \dots, \lambda^{(l-1)}) \mapsto (\lambda^{(1)}, \dots, \lambda^{(l-1)}, \lambda^{(0)}).$$

(2) *The crystal isomorphism $B(\mathbf{v}) \to B(\sigma_j\mathbf{v})$ is given by*

$$(\lambda^{(0)}, \dots, \lambda^{(j-1)}, \lambda^{(j)}, \dots, \lambda^{(l-1)}) \mapsto (\lambda^{(0)}, \dots, \tilde{\lambda}^{(j)}, \tilde{\lambda}^{(j-1)}, \dots, \lambda^{(l-1)}),$$

where, $\tilde{\lambda}^{(j-1)}$ and $\tilde{\lambda}^{(j)}$ are defined by

$$\lambda^{(j)} \otimes \lambda^{(j-1)} \mapsto \tilde{\lambda}^{(j-1)} \otimes \tilde{\lambda}^{(j)}$$

under the following crystal isomorphism, called a combinatorial R-matrix, between $\mathfrak{g}(A_\infty)$-crystals.

$$B(\Lambda_{\mathrm{v}_j}) \otimes B(\Lambda_{\mathrm{v}_{j-1}}) \to B(\Lambda_{\mathrm{v}_{j-1}}) \otimes B(\Lambda_{\mathrm{v}_j}).$$

The combinatorial R-matrix may be computed in a purely combinatorial manner. See [13] for the details.

5.5. **Crystal embedding to $B(\infty)$.** Let $T_\Lambda = \{t_\Lambda\}$ be the crystal defined by $\mathrm{wt}(t_\Lambda) = \Lambda$, $\epsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$ and $\tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = 0$. Then, by the theory of crystals, we have the crystal embedding $B(\Lambda) \hookrightarrow B(\infty) \otimes T_\Lambda$ such that

(i) the image of the embedding is given by
$$\{b \otimes t_\Lambda \in B(\infty) \otimes T_\Lambda \mid \epsilon_i(b^*) \le \Lambda(\alpha_i^\vee)\},$$
where $b \mapsto b^*$ is the involution on $B(\infty)$ which is induced by the anti-automorphism of U_v^- defined by $f_i \mapsto f_i$,

(ii) $b \otimes t_\Lambda$ belongs to the image if and only if $G_v(b)v_\Lambda \neq 0$, where v_Λ is the highest weight vector of $V_v(\Lambda)$.

We identify $B(\infty)$ with the crystal of aperiodic multisegments defined in Theorem 2.12 and used in Theorem 4.4. As $B(\mathbf{v})$ is isomorphic to $B(\Lambda)$, we have the crystal embedding
$$B(\mathbf{v}) \hookrightarrow B(\infty) \otimes T_\Lambda$$
in the language of Uglov l-partitions and multisegments.

We shall describe this embedding in subsequent subsections. By virtue of Theorem 5.7, we may assume that $\mathbf{v} \in \mathcal{V}_l$. Write the crystal embedding by $\boldsymbol{\lambda} \mapsto f(\boldsymbol{\lambda}) \otimes t_\Lambda$, and denote both the empty l-partition and the empty multisegment by the common symbol $\emptyset$. Then, the crystal embedding sends $\emptyset$ to $\emptyset \otimes t_\Lambda$, and the tensor product rule shows that for any path
$$\emptyset \xrightarrow{i_1} \boldsymbol{\lambda}_1 \xrightarrow{i_2} \boldsymbol{\lambda}_2 \xrightarrow{i_3} \cdots \xrightarrow{i_n} \boldsymbol{\lambda}_n$$
in $B(\mathbf{v})$, we have the corresponding path
$$\emptyset \xrightarrow{i_1} f(\boldsymbol{\lambda}_1) \xrightarrow{i_2} f(\boldsymbol{\lambda}_2) \xrightarrow{i_3} \cdots \xrightarrow{i_n} f(\boldsymbol{\lambda}_n)$$
in $B(\infty)$, and vice versa. On the other hand, if one can prove this property for some map $f : B(\mathbf{v}) \to B(\infty)$ then it follows that
$$\epsilon_i(\boldsymbol{\lambda}) = \epsilon_i(f(\boldsymbol{\lambda}) \otimes t_\Lambda) \ \text{ and } \ \mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(f(\boldsymbol{\lambda}) \otimes t_\Lambda),$$
so that we also have $\varphi_i(\boldsymbol{\lambda}) = \varphi_i(f(\boldsymbol{\lambda}) \otimes t_\Lambda)$. Hence, we may conclude that the map $\boldsymbol{\lambda} \mapsto f(\boldsymbol{\lambda}) \otimes t_\Lambda$ is a crystal embedding in the sense of [14] and it must coincide with the crystal embedding $B(\mathbf{v}) \hookrightarrow B(\infty) \otimes T_\Lambda$.

5.6. **Row lengths and the order $\prec_{\mathbf{v}}$.** We prove two lemmas which relate the length of rows of an l-partition and the order $\prec_{\mathbf{v}}$.

Lemma 5.8. *Let $\mathbf{v} \in \mathcal{V}_l$ and $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(l-1)}) \in \Phi(\mathbf{v})$. Suppose that $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ are i-nodes of $\boldsymbol{\lambda}$ such that each node is either addable or removale i-node. Then, $\lambda^{(c_1)}_{a_1} < \lambda^{(c_2)}_{a_2}$ implies $\gamma_1 \prec_{\mathbf{v}} \gamma_2$.*

Proof. We show that $\gamma_2 \preceq_{\mathbf{v}} \gamma_1$ implies $\lambda^{(c_1)}_{a_1} \ge \lambda^{(c_2)}_{a_2}$. As an intermediate step, we first claim that $\gamma_2 \preceq_{\mathbf{v}} \gamma_1$ implies $\lambda^{(c_1)}_{a_1} \ge \lambda^{(c_2)}_{b_1-b_2+a_2}$. Note that we have $c(\gamma_1) \ge c(\gamma_2)$ by $\gamma_2 \preceq_{\mathbf{v}} \gamma_1$. Hence, we have
$$a_1 \le b_1 - b_2 + a_2 + \mathrm{v}_{c_1} - \mathrm{v}_{c_2},$$

which implies $\lambda_{a_1}^{(c_1)} \geq \lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1}-\mathrm{v}_{c_2}}^{(c_1)}$.

Suppose that $c_1 \leq c_2$. As $\boldsymbol{\lambda}$ is a FLOTW l-partition, we have

$$\lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1}-\mathrm{v}_{c_2}}^{(c_1)} \geq \lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1+1}-\mathrm{v}_{c_2}}^{(c_1+1)} \geq \cdots \geq \lambda_{b_1-b_2+a_2}^{(c_2)}.$$

Hence $\lambda_{a_1}^{(c_1)} \geq \lambda_{b_1-b_2+a_2}^{(c_2)}$ follows.

Suppose that $c_1 > c_2$. Then, $c(\gamma_1) > c(\gamma_2)$ and we must have

$$b_1 - a_1 + \mathrm{v}_{c_1} \geq b_2 - a_2 + \mathrm{v}_{c_2} + e,$$

because γ_1 and γ_2 have the same residue i. Hence, we have

$$a_1 \leq b_1 - b_2 + a_2 + \mathrm{v}_{c_1} - \mathrm{v}_{c_2} - e,$$

which implies $\lambda_{a_1}^{(c_1)} \geq \lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1}-\mathrm{v}_{c_2}-e}^{(c_1)}$. Then, by the same reasoning as above, we have

$$\begin{aligned}\lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1}-\mathrm{v}_{c_2}-e}^{(c_1)} &\geq \lambda_{b_1-b_2+a_2+\mathrm{v}_{c_1+1}-\mathrm{v}_{c_2}-e}^{(c_1+1)} \geq \cdots \\ &\geq \lambda_{b_1-b_2+a_2+\mathrm{v}_{l-1}-\mathrm{v}_{c_2}-e}^{(l-1)} \geq \lambda_{b_1-b_2+a_2+\mathrm{v}_0-\mathrm{v}_{c_2}}^{(0)} \geq \cdots \geq \lambda_{b_1-b_2+a_2}^{(c_2)}.\end{aligned}$$

Hence $\lambda_{a_1}^{(c_1)} \geq \lambda_{b_1-b_2+a_2}^{(c_2)}$ follows again.

If $b_1 \leq b_2$ then $b_1-b_2+a_2 \leq a_2$ implies the desired inequality $\lambda_{a_1}^{(c_1)} \geq \lambda_{a_2}^{(c_2)}$. Suppose that $b_1 > b_2$. As γ_1 is either addable or removable i-node, we have either $b_1 = \lambda_{a_1}^{(c_1)} + 1$ or $b_1 = \lambda_{a_1}^{(c_1)}$. Similarly, we have either $b_2 = \lambda_{a_2}^{(c_2)} + 1$ or $b_2 = \lambda_{a_2}^{(c_2)}$. Hence, we have $\lambda_{a_1}^{(c_1)} \geq b_1 - 1 \geq b_2 \geq \lambda_{a_2}^{(c_2)}$. □

Lemma 5.9. *Let $\boldsymbol{\lambda}$ be a FLOTW l-partition, and let $\gamma_A = (a', b+1, c')$ and $\gamma_R = (a, b, c)$ be addable and removable i-nodes of $\boldsymbol{\lambda}$ respectively. Then we have $\gamma_R \prec_{\mathbf{v}} \gamma_A$.*

Proof. Suppose to the contrary that $\gamma_A \prec_{\mathbf{v}} \gamma_R$. Then we have either

(i) $c(\gamma_A) < c(\gamma_R)$, or
(ii) $c(\gamma_A) = c(\gamma_R)$ and $c' > c$.

In case (i), $b - a + \mathrm{v}_c \geq b + 1 - a' + \mathrm{v}_{c'} + e$ so that $a + \mathrm{v}_{c'} - \mathrm{v}_c + e \leq a' - 1$. As γ_A is an addable node, we also have $\lambda_{a'-1}^{(c')} > \lambda_{a'}^{(c')}$. Then, $a + \mathrm{v}_{c'} - \mathrm{v}_c + e < a'$ implies that

$$\lambda_{a+\mathrm{v}_{c'}-\mathrm{v}_c}^{(c')} \geq \lambda_{a+\mathrm{v}_{c'}-\mathrm{v}_c+e}^{(c')} > \lambda_{a'}^{(c')}.$$

Now, using the assumption that $\boldsymbol{\lambda}$ is a FLOTW l-partition, we have

$$\begin{cases} \lambda_a^{(c)} \geq \lambda_{a+\mathrm{v}_{c'}-\mathrm{v}_c}^{(c')} > \lambda_{a'}^{(c')} & \text{if } c \leq c', \\ \lambda_a^{(c)} \geq \lambda_{a+\mathrm{v}_{c'}-\mathrm{v}_c+e}^{(c')} > \lambda_{a'}^{(c')} & \text{if } c > c'. \end{cases}$$

However, $\lambda_a^{(c)} = b$ since γ_R is a removable node, and $\lambda_{a'}^{(c')} = b$ since γ_A is an addable node. Thus, we have reached a contradiction.

In case (ii), $b-a+\mathrm{v}_c = b+1-a'+\mathrm{v}_{c'}$ implies $a+\mathrm{v}_{c'}-\mathrm{v}_c+1=a'$. As γ_A is an addable node, $\lambda^{(c')}_{a+\mathrm{v}_{c'}-\mathrm{v}_c} > \lambda^{(c')}_{a'}$. Thus, $c'>c$ implies that

$$\lambda^{(c)}_a \geq \lambda^{(c')}_{a+\mathrm{v}_{c'}-\mathrm{v}_c} > \lambda^{(c')}_{a'}.$$

However, we have $\lambda^{(c)}_a = b$ and $\lambda^{(c')}_{a'} = b$ as before, so that we have reached a contradiction again. □

5.7. **The map $f_{\mathbf{v}}$.** For each FLOTW l-partition $\boldsymbol{\lambda} \in \Phi(\mathbf{v})$, we associate a multisegment which is a collection of segments

$$[1-i+\mathrm{v}_c; \lambda^{(c)}_i),$$

where $\lambda^{(c)}_i$ are parts of $\lambda^{(c)}$, for $c=0,\dots,l-1$. This defines a well-defined map $f_{\mathbf{v}} : \Phi(\mathbf{v}) \to B(\infty)$.

Example 5.10. Let $e=4$, and let $\boldsymbol{\lambda} = ((2,1),(1)) \in \Phi((0,1))$. Then

$$f_{(0,1)}(\boldsymbol{\lambda}) = \{[0,1],[3],[1]\}.$$

Next let $\boldsymbol{\lambda} = ((2),(1),(1)) \in \Phi((0,1,3))$. Then we have the same result

$$f_{(0,1,3)}(\boldsymbol{\lambda}) = \{[0,1],[1],[3]\}.$$

Then we may prove the following. Note that the fact itself was observed by several people including the first author years ago, but the authors do not know any reference which proves this.

Theorem 5.11. *Suppose that $\mathbf{v} \in \mathcal{V}_l$. Then, the crystal embedding $\Phi(\mathbf{v}) \hookrightarrow B(\infty) \otimes T_\Lambda$ is given by $\boldsymbol{\lambda} \mapsto f_{\mathbf{v}}(\boldsymbol{\lambda}) \otimes t_\Lambda$.*

Proof. As was explained in the previous subsection, it suffices to show that there is an arrow

$$\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$$

in $B(\mathbf{v})$ if and only if there is an arrow

$$f_{\mathbf{v}}(\boldsymbol{\lambda}) \xrightarrow{i} f_{\mathbf{v}}(\boldsymbol{\mu})$$

in $B(\infty)$.

We read the addable and removable i-nodes of $\boldsymbol{\mu}$ in increasing order with respect to the total order $\prec_{\mathbf{v}}$. Let $\gamma_1 \dots \gamma_m$ be the resulting word of the nodes. On the other hand, we read the same set of addable and removable i-nodes of $\boldsymbol{\mu}$ in increasing order with respect to the length of the corresponding segments in $f_{\mathbf{v}}(\boldsymbol{\mu})$. If the length are the same, we declare that removable i-nodes precede addable i-nodes. We denote the resulting word $\gamma_{\sigma(1)} \cdots \gamma_{\sigma(m)}$, for $\sigma \in \mathfrak{S}_m$.

Write $\gamma_j = (a_j, b_j, c_j)$, for $1 \leq j \leq m$. Then, Lemma 5.8 implies that if $\lambda^{(c_j)}_{a_j} \neq \lambda^{(c_k)}_{a_k}$ then $j<k$ implies $\sigma^{-1}(j) < \sigma^{-1}(k)$. On the other hand, Lemma 5.9 implies that if $\lambda^{(c_j)}_{a_j} = \lambda^{(c_k)}_{a_k}$ then $j<k$ implies $\sigma^{-1}(j) < \sigma^{-1}(k)$. We conclude that σ is the identity.

We define $S'_{k,i}$ to be the number of addable i-nodes minus the number of removable i-nodes in $\{\gamma_k, \gamma_{k+1}, \dots, \gamma_m\}$.

Suppose that $\tilde{e}_i \boldsymbol{\mu} = \boldsymbol{\lambda}$ and let $\gamma = (a, b, c)$ be the good i-node of $\boldsymbol{\mu}$. Then $\min_{k>0} S'_{k,i}$ is attained at γ. Define k_r, for $r > 0$, by

$$k_r = \min\{j \mid \lambda_{a_j}^{(c_j)} \geq r\}.$$

It is clear that $\min_{k>0} S'_{k,i}$ is attained only at removable nodes of the form γ_{k_r}, for some r. Now observe that addable and removable i-nodes of the multisegment $f_{\mathbf{v}}(\boldsymbol{\mu})$ which do not belong to $\{\gamma_1, \dots, \gamma_m\}$ come from pairs of consecutive rows of the same length in $\boldsymbol{\mu}$. Let $m_{(k;i]}$ be the multiplicity of $(k;i]$ in $f_{\mathbf{v}}(\boldsymbol{\mu})$. Then, by the above observation, we have

$$S_{r,i} = \sum_{k \geq r} (m_{(k,i-1]} - m_{(k;i]}) = S'_{k_r,i},$$

and $\min_{r>0} S_{r,i}$ is attained at $r = b$. Instead of proving that b is the unique r that attains the minimum, we shall show that $\tilde{f}_i f_{\mathbf{v}}(\boldsymbol{\lambda}) = f_{\mathbf{v}}(\boldsymbol{\mu})$. As γ is the good removable i-node of $\boldsymbol{\mu}$, the following is clear.

> If $r < b$ then, among the nodes γ_j, for $k_r \leq j < k_b$, the number of addable nodes is always greater than or equal to the number of removable nodes.

This implies that, if we change the status of γ from a removable node to an addable node, then $S_{r,i} > S_{b,i}$ if $r \leq b - 1$, for the new values $S_{r,i}$ and $S_{b,i}$ computed after we change the status of γ. If we consider normal i-nodes which appear to the right of γ, it is also clear that $S_{r,i} \geq S_{b,i}$ if $r \geq b + 1$, for the new values $S_{r,i}$. Thus, we obtain $\tilde{f}_i f_{\mathbf{v}}(\boldsymbol{\lambda}) = f_{\mathbf{v}}(\boldsymbol{\mu})$.

Next suppose that $\tilde{f}_i f_{\mathbf{v}}(\boldsymbol{\lambda}) = f_{\mathbf{v}}(\boldsymbol{\mu})$. We consider $S_{r,i}$ and suppose that $\min_{r>0} S_{r,i}$ is attained at $\ell_0 < \ell_1 < \cdots$. The minimum value is attained at a removable i-node which is the leftmost node among the nodes of the segments of the same length. Then, the minimality implies that the right neighbor of the removable node is addable. We denote this node by γ. We show that γ is the good addable i-node of $\boldsymbol{\lambda}$.

Suppose that γ is cancelled in the RA-deletion procedure. If the removable i-node which cancels R is not of the form γ_{k_r}, it contradicts the minimality of $S_{\ell_0,i}$. Thus, the removable node is γ_{k_b}, for some $b < \ell_0$. Then, $S_{b,i} = S_{\ell_0,i}$ implies $\ell_0 \leq b$, which contradicts $b < \ell_0$. Hence, we have proved that γ is a normal addable i-node. If there was another normal addable i-node to the right of γ, it would contradict the minimality of $S_{\ell_0,i}$, so that γ is the good addable i-node of $\boldsymbol{\lambda}$. Thus, we obtain $\tilde{f}_i \boldsymbol{\lambda} = \boldsymbol{\mu}$. □

Define $B^{\mathrm{ap}}(\Lambda) = \{\psi \in B(\infty) \mid \epsilon_i(\psi^*) \leq \Lambda(\alpha_i^\vee)\}$. As we have proved that $\boldsymbol{\lambda} \mapsto f_{\mathbf{v}}(\boldsymbol{\lambda}) \otimes t_\Lambda$ is the crystal embedding $B(\Lambda) \hookrightarrow B(\infty) \otimes T_\Lambda$ in the language of FLOTW and multisegment realizations, we have the following corollary. The basis in Corollary 5.12(2) is the *canonical basis* of $V_v(\Lambda)$. The statement is for the crystal structure we have chosen, but it is easy to state it for the other, since the Kashiwara involution on the set of aperiodic multisegments is explicitly described in [12].

Corollary 5.12.
(1) $f_{\mathbf{v}}(B(\mathbf{v})) = B^{\mathrm{ap}}(\Lambda)$.
(2) $\{G_v(\psi)v_\Lambda \mid \psi \in B^{\mathrm{ap}}(\mathrm{v})\}$ *is a basis of* $V_v(\Lambda)$.

6. Fock space theory for cyclotomic Hecke algebras

In this section, we give the combinatorial proof of the modular branching rule. The proof depends on Lemma 6.7, which says that isomorphisms of crystals give the correspondence of labels of a simple $\mathcal{H}_n^\Lambda$-module, which is labelled by various realizations of the crystal $B(\Lambda)$. Hence, the explicit description of the isomorphisms in the previous section gives us the module correspondence.

6.1. **Cyclotomic Hecke algebras.** Let $\mathbf{v}$ be a multicharge as before. The *cyclotomic Hecke algebra* $\mathcal{H}_n^{\mathbf{v}}(q)$ is the quotient algebra $H_n/I_{\mathbf{v}}$ of the affine Hecke algebra H_n, where $I_{\mathbf{v}}$ is the ideal of H_n generated by the polynomial $\prod_{i=0}^{l-1}(X_1 - q^{\mathrm{v}_i})$. If we specialize $q = \zeta$, the algebra depends only on Λ, and we denote the algebra by $\mathcal{H}_n^\Lambda$. This is the main object of the study in the remaining part of the paper. As $\mathcal{H}_n^\Lambda$ is a quotient algebra of the affine Hecke algebra H_n, the set of simple $\mathcal{H}_n^\Lambda$-modules is a subset of simple H_n-modules. In fact, by Fock space theory for cyclotomic Hecke algebras we will explain in the next subsection, we know that it is the set $\{D_\psi \mid \psi \in B^{\mathrm{ap}}(\Lambda)\}$.

Definition 6.1. We denote by $\mathcal{H}_n^\Lambda$ -mod the category of finite-dimensional $\mathcal{H}_n^\Lambda$-modules.

Note that $\mathcal{H}_n^{\mathbf{v}}(q)$ is a cellular algebra in the sense of Graham and Lehrer: it has the Specht module theory developped by Dipper, James and Mathas. Then, the first author showed that simple $\mathcal{H}_n^\Lambda$-modules are labelled by Kleshchev l-partitions. We refer to [2, Ch. 12] for details.

For $\boldsymbol{\lambda} \in \Phi^K$, we denote by $D^{\boldsymbol{\lambda}}$ the simple $\mathcal{H}_n^\Lambda$-module labelled by $\boldsymbol{\lambda}$. For $\boldsymbol{\lambda} \in \Phi(\mathbf{v})_n$, we define $\widetilde{D}^{\boldsymbol{\lambda}}$ by

$$\widetilde{D}^{\boldsymbol{\lambda}} = D^{\Gamma(\boldsymbol{\lambda})}.$$

We will explain in the next subsection that this labelling coincides with the Geck-Rouquier-Jacon parametrization of simple $\mathcal{H}_n^\Lambda$-modules in terms of the canonical basic set.

Before giving the second proof, we complete the first proof. Namely, we prove Theorem 6.2 below, which compares the geometrically defined simple $\mathcal{H}_n^\Lambda$-modules and the combinatorially defined simple $\mathcal{H}_n^\Lambda$-modules by using Theorem 4.4.

Theorem 6.2. *Let* $\boldsymbol{\lambda}$ *be an* l*-partition. Then,* $\widetilde{D}^{\boldsymbol{\lambda}} \simeq D_{f_{\mathbf{v}}(\boldsymbol{\lambda})}$ *as* H_n*-modules.*

Proof. We have $i\text{-}\operatorname{Res}(D_\psi) \simeq D_{\tilde{e}_i\psi}$ by Theorem 4.4. On the other hand, we have $i\text{-}\operatorname{Res}(D^{\boldsymbol{\lambda}}) \simeq D^{\tilde{e}_i\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \Phi^K$, in [3, Theorem 6.1]. Note that if $i\text{-}\operatorname{Res}(D^{\boldsymbol{\lambda}}) \simeq i\text{-}\operatorname{Res}(D_\psi) \neq 0$ then $D^{\boldsymbol{\lambda}} \simeq D_\psi$. This property of crystals is a consequence of the Frobenius reciprocity. Hence, we may prove the claim by induction on n. □

6.2. **Standard modules.** We say a few words on the standard modules of the affine Hecke algebra. Let $X \in \mathcal{O}_\psi$ and consider

$$(\mathcal{F}\ell_n^a)_X = \{F \in \mathcal{F}\ell_n^a \mid XF_i \subseteq F_{i-1}\}.$$

Then, $H_*((\mathcal{F}\ell_n^a)_X, \mathbb{C})$ is an $H_*^{BM}(Z_n^a, \mathbb{C})$-module by the convolution action, and it is called the *standard module*. We denote it by M_ψ. Suppose that X is a principal nilpotent element so that $\psi = [i; l)$ for some $i \in \mathbb{Z}/e\mathbb{Z}$ and $l \in \mathbb{Z}_{>0}$. Then, $(\mathcal{F}\ell_n^a)_X$ is a point, which is the flag

$$0 \subseteq \operatorname{Ker}(X) \subseteq \operatorname{Ker}(X^2) \cdots \subseteq \operatorname{Ker}(X^n) = V$$

of flag type $(i+l-1, \dots, i+1, i)$, and the proof of Lemma 4.6 shows that, if we follow the identification $H_n \simeq K^{G_n \times \mathbb{C}^\times}(Z_n)$ in [22], then M_ψ is the one dimensional H_n-module given by $T_i \mapsto -1$ and

$$X_1 \mapsto \zeta^{i+l-1}, \dots, X_{n-1} \mapsto \zeta^{i+1}, X_n \mapsto \zeta^i.$$

Thus, M_ψ for general ψ coincides with the induced up module of the tensor product of such one dimensional modules over the affine Hecke algebras associated with segments in ψ, in the Grothendieck group of the module category of the affine Hecke algebra.

Now, we switch to the other identification used in Theorem 4.4. Define the *standard module* N_ψ by

$$N_\psi = {}^\sigma M_{\rho(\psi)}.$$

Then N_ψ is given by $T_i \mapsto \zeta$ and

$$X_1 \mapsto \zeta^{i-l+1}, \dots, X_{n-1} \mapsto \zeta^{i-1}, X_n \mapsto \zeta^i.$$

when $\psi = (l; i]$. This is the standard module in [2]. Then, a key observation used in [2] was the equality

$$G_{v=1}(\psi) = \sum_{\psi'} [N_{\psi'} : D_\psi] u_{\psi'}$$

in the Hall algebra in subsection 2.3 evaluated at $v = 1$. [5] Now we are able to give an example of Theorem 6.2.

[5]In fact, the choice of the identification played no role in [2] because it sufficed for us to prove the statement that the canonical basis evaluated at $v = 1$ coincides with the dual basis of simples in the Fock space, and we did not need compare individual simple modules.

Example 6.3. Let $e = 3$. Then, we have

$$G_{v=1}(\{(2;2]\}) = u_{\{(2;2]\}} + u_{\{(1;1],(1;2]\}}, \quad G_{v=1}(\{(1;1],(1;2]\}) = u_{\{(1;1],(1;2]\}}.$$

Note that $N_{\{(1;1]\}}$ and $N_{\{(1;2]\}}$ are one dimensional H_1-modules defined by $X_1 \mapsto \zeta$ and $X_1 \mapsto \zeta^2$, respectively. Then, $N_{\{(2;2]\}} = D_{\{(2;2]\}}$ is the simple module defined by

$$X_1 \mapsto \zeta, \quad X_2 \mapsto \zeta^2, \quad T_1 \mapsto \zeta,$$

and $N_{\{(1;1],(1;2]\}}$ is the module induced from $N_{\{(1;1]\}} \otimes N_{\{(1;2]\}}$. Thus, we deduce that $D_{\{(1;1],(1;2]\}}$ is the simple module defined by

$$X_1 \mapsto \zeta^2, \quad X_2 \mapsto \zeta, \quad T_1 \mapsto -1.$$

(Ex.1) Suppose that $l = 1$. Then we have

for $\mathbf{v} = 1$ $D^{(2)} \simeq D_{\{(2;2]\}}$ and for $\mathbf{v} = 2$ $D^{(1^2)} \simeq D_{\{(1;1],(1;2]\}}$,

with $(2), (1^2) \in \Phi_2^K$. This follows from the explicit construction of Specht modules. Since $(2) = \tilde{f}_2\tilde{f}_1\emptyset$ and $(1^2) = \tilde{f}_1\tilde{f}_2\emptyset$ in Φ_2^K, we have $\Gamma((2)) = (2)$ and $\Gamma((1^2)) = (1^2)$, so that

$$\widetilde{D}^{(2)} \simeq D_{\{(2;2]\}} \text{ and } \widetilde{D}^{(1^2)} \simeq D_{\{(1;1],(1;2]\}}.$$

(Ex.2) Suppose that $l = 2$ and $\mathbf{v} = (1,2)$. Then, $\tilde{f}_2\tilde{f}_1\emptyset = ((2),\emptyset)$ and $\tilde{f}_1\tilde{f}_2\emptyset = ((1),(1))$ in $\Phi(\mathbf{v})$, so that

$$\widetilde{D}^{((2),\emptyset)} \simeq D_{\{(2;2]\}} \text{ and } \widetilde{D}^{((1),(1))} \simeq D_{\{(1;1],(1;2]\}}.$$

6.3. **Fock space theory.** In this subsection, we explain the Fock space theory for cyclotomic Hecke algebras. In the following, $G_v(b)$, U_v^-, etc. at $v = 1$ are denoted by $G(b)$, U^-, etc.

Let $\mathcal{C}_n$ be the full subcategory of H_n-mod consisting of finite dimensional H_n-modules on which $X_1, \ldots, X_n$ have eigenvalues in $\{1, \zeta, \ldots, \zeta^{e-1}\}$.

Definition 6.4. Let

$$U_n = \mathrm{Hom}_{\mathbb{C}}(K_0(\mathcal{C}_n), \mathbb{C}) \text{ and } V_n = \mathrm{Hom}_{\mathbb{C}}(K_0(\mathcal{H}_n^\Lambda\text{-mod}), \mathbb{C})$$

be the dual spaces of the Grothendieck groups of $\mathcal{C}_n$ and $\mathcal{H}_n^\Lambda$-mod, and define

$$U = \bigoplus_{n \geq 0} U_n \text{ and } V = \bigoplus_{n \geq 0} V_n.$$

Hereafter, we identify V_n with the split Grothendieck group of the additive subcategory of $\mathcal{H}_n^\Lambda$-mod consisting of projective $\mathcal{H}_n^\Lambda$-modules.

U_n has the dual basis

$$\{[D_\psi]^* \mid \psi \text{ is an aperiodic multisegment of rank } n.\}$$

which is dual to the basis consisting of simple $\mathcal{H}_n^\Lambda$-modules.

Let $\pi : U \to V$ be the natural map and define

$$p : U^- \to V(\Lambda) \subseteq \mathcal{F}$$

by $F \mapsto Fv_\Lambda$, for $F \in U^-$.

The theorem below states the most basic result in the Fock space theory. See [1] or [2, Theorem 14.49].

Theorem 6.5.
(1) *U has structure of a U^--module and V has structure of a $\mathfrak{g}$-module.*
(2) *U is isomorphic to the regular representation of U^- such that*

$$[D_\psi]^* \mapsto G(\psi).$$

(3) *V is isomorphic to $V(\Lambda)$ and the basis*

$$\bigsqcup_{n\geq 0} \{[P] \mid P \text{ is an indecomposable } \mathcal{H}_n^\Lambda\text{-module.}\}$$

of V corresponds to the canonical basis of $V(\Lambda)$ under the isomorphism.
(4) *The following diagram commutes:*

$$\begin{array}{ccc} U & \simeq & U^- \\ \pi\downarrow & & \downarrow p \\ V & \simeq & V(\Lambda) \end{array}$$

6.4. **The combinatorial proof.** First we make it clear what we mean by "simple $\mathcal{H}_n^\Lambda$-modules are labelled by Uglov l-partitions".

Definition 6.6. We say that *simple $\mathcal{H}_n^\Lambda$-modules are labelled by $B(\mathbf{v})$*, if the projective cover of a simple $\mathcal{H}_n^\Lambda$-module is equal to $G(\boldsymbol{\lambda}) \in \mathcal{F}^{\mathbf{v}}$ in Theorem 6.5(3), for $\boldsymbol{\lambda} \in B(\mathbf{v})$, then the label of the simple module is $\boldsymbol{\lambda}$.

It is proved by the first author that Specht module theory is an example of the statement that simple $\mathcal{H}_n^\Lambda$-modules are labelled by $B(\mathbf{v})$. Another example is provided by the second author. Recall that Geck and Rouquier invented different theory to label simple modules by using Lusztig's a-values. The labelling set is called the *canonical basic set*. When we work with Hecke algebras of type B, it provides us with a set of bipartitions. The second author has generalized the theory to cyclotomic Hecke algebras and his result says that simple $\mathcal{H}_n^\Lambda$-modules are labelled by $\Phi(\mathbf{v})$, for $\mathbf{v} \in \mathcal{V}_l$.

If one uses Theorem 6.5, it is quite easy to identify simple $\mathcal{H}_n^\Lambda$-modules in various labellings.

Lemma 6.7.
(1) *Suppose that simple $\mathcal{H}_n^\Lambda$-modules are labelled by $B(\mathbf{v})$. Let*

$$f_{\mathbf{v},\infty} : B(\mathbf{v}) \simeq B^{\mathrm{ap}}(\Lambda) \subseteq B(\infty)$$

be the unique crystal isomorphism. Then, $D^{\boldsymbol{\lambda}} \simeq D_{f_{\mathbf{v}}(\boldsymbol{\lambda})}$ as H_n-modules.
(2) *For two labelling $B(\mathbf{v})$ and $B(\mathbf{w})$ of simple $\mathcal{H}_n^\Lambda$-modules, we denote the set of simple modules by*

$$\{D_{\mathbf{v}}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in B(\mathbf{v})\} \text{ and } \{D_{\mathbf{w}}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in B(\mathbf{v})\},$$

respectively. Let $f_{\mathbf{v},\mathbf{w}} : B(\mathbf{v}) \simeq B(\mathbf{w})$ be the unique crystal isomorphism. Then, $D_{\mathbf{v}}^{\boldsymbol{\lambda}} \simeq D_{\mathbf{w}}^{f_{\mathbf{v},\mathbf{w}}(\boldsymbol{\lambda})}$ as H_n-modules.

Proof. (1) Suppose that $f_{\mathbf{v},\infty}(\boldsymbol{\lambda}) = \psi$. Then, we have $G_v(\psi)\emptyset = G_v(\boldsymbol{\lambda})$. Specializing at $v = 1$, we obtain $G(\psi) = P^{\boldsymbol{\lambda}}$. Then, using the commutativity of the diagram in Theorem 6.5(4), we conclude that $\pi([D_\psi]^*) = [D^{\boldsymbol{\lambda}}]^*$, which is identified with $P^{\boldsymbol{\lambda}}$. Hence, $D_\psi \simeq D^{\boldsymbol{\lambda}}$ as H_n-modules.

(2) First we apply (1) to two crystal isomorphisms $B(\mathbf{v}) \simeq B^{ap}(\Lambda)$ and $B^{\mathrm{ap}}(\Lambda) \simeq B(\mathbf{w})$. Then use the fact that $f_{\mathbf{v},\mathbf{w}} = f_{\mathbf{w},\infty}^{-1} \circ f_{\mathbf{v},\infty}$. □

As we have established Lemma 6.7, we can derive the modular branching rule for the affine Hecke algebra from this.

Theorem 6.8. *For each aperiodic multisegment ψ, we have*

$$\mathrm{Soc}(i\text{-}\mathrm{Res}_{H_{n-1}}^{H_n}(D_\psi)) \simeq D_{\tilde{e}_i\psi}.$$

Proof. Choose Λ sufficiently large so that $f_{\mathbf{v}}(B(\mathbf{v})) = B^{ap}(\Lambda)$ may contain any path

$$\emptyset \xrightarrow{i_1} \psi_1 \xrightarrow{i_2} \psi_2 \xrightarrow{i_3} \cdots \xrightarrow{i_n} \psi_n = \psi$$

in $B(\infty)$ from $\emptyset$ to ψ. Let $i \in \mathbb{Z}/e\mathbb{Z}$ be such that $\tilde{e}_i\psi \neq 0$ and let $\boldsymbol{\lambda} \in B(\mathbf{v})$ be such that $f_{\mathbf{v}}(\boldsymbol{\lambda}) = \psi$. Then $\tilde{e}_i\boldsymbol{\lambda} \neq 0$ and $f_{\mathbf{v}}(\tilde{e}_i\boldsymbol{\lambda}) = \tilde{e}_i\psi$. Then, the previous Lemma yields the isomorphisms

$$D_\psi \simeq \widetilde{D}^{\boldsymbol{\lambda}} \text{ and } D_{\tilde{e}_i\psi} \simeq \widetilde{D}^{\tilde{e}_i\boldsymbol{\lambda}}.$$

Thus,

$$\mathrm{Soc}(i\text{-}\mathrm{Res}_{H_{n-1}}^{H_n}(D_\psi)) \simeq \mathrm{Soc}(i\text{-}\mathrm{Res}_{H_{n-1}}^{H_n}(\widetilde{D}^{\boldsymbol{\lambda}})) \simeq \widetilde{D}^{\tilde{e}_i\boldsymbol{\lambda}} \simeq D_{\tilde{e}_i\psi},$$

where the middle isomorphism is the modular branching rule in the labelling by Kleshchev l-partitions [3, Theorem 6.1]. We have proved the theorem. □

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S.A.: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: ariki@kurims.kyoto-u.ac.jp

N.J.:Université de Franche-Comté, UFR Sciences et Techniques, 16 route de Gray, 25 030 Besançon, France.
E-mail address: njacon@univ-fcomte.fr

C. L.: Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville Centre Universitaire de la Mi-Voix B.P. 699 62228 Calais France
E-mail address: Cedric.Lecouvey@lmpa.univ-littoral.fr